Some aspects of the eigenfrequency computation in a two-dimensional tank filled with two non miscible fluids.

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Highlights

- the eigenfreqencies of the sloshing modes of a two fluid layer system are computed,
- the convergence of a desingularized technique is commented,
- the variations of the eigenfrequencies with the density ratio are detailed.

1) Introduction

Determining the natural frequencies of the sloshing modes in a two-dimensional tank is a well-known problem in potential theory (see Ibrahim, 2005, or Faltinsen and Timokha, 2009, among recent textbooks on that topic). In practice the computational effort reduces to the determination of eigenvalues of a matrix. A first (and non sophisticated) approach consists in formulating a classical integral equation where singularities are distributed all over the fluid boundaries, including the tank walls and the free surface itself. On the former surface, a homogeneous Neumann condition is prescribed whereas on the latter surface a Fourier condition is prescribed. Then the matrix follows from the discretization of the surface integrals into panels. An improvement is to use harmonic solutions which implicitely account for the wall boundary condition. Faltinsen and Timokha (2010, 2012, 2014) deal with the case of the circular tank or prismatic tank in that way, since they need an analytic continuation above the linearized free surface. This is obviously required when the fully nonlinear free surface flow problem is solved. This is why the technique initially proposed by Tuck (1998) is appealing. A conformal mapping turns the domain bounded by the tank walls into a half space and a desingularized technique provides a simple expression of the velocity potential. The computational effort then focuses on the free surface description only. That combination has proved to be very effective and robust even for more complicated tank geometries than a simple rectangular tank (see Scolan, 2010). The same technique can be used to deal with a two fluid layer system in a closed tank with a more or less complicated boundaries (see Scolan *et al*, 2014).

This technique and other analytical approaches are detailed in the next developments in order to determine the frequencies of the sloshing modes of a two fluid layer system in a closed tank. First a numerical analysis is performed for a rectangular tank to assess the convergence criteria in terms of the number of singularities and the desingularizing distance. Then the technique proposed by McIver (1989) is applied to a two fluid layer system in a circular tank.

2) Eigenfrequency in a rectangular tank by MFS

Much work has been done for a single phase flow neglecting the gas above the free surface, considered as vacuum. However in most practical situations there is a gas above the liquid. From the analysis of Lord Rayleigh (1883) in potential theory, the dispersion relation of wave travelling at the interface of two non miscible fluids in an unbounded fluid domain, is well-known. This dispersion relation reads $\omega^2 = gkA_t$ where ω is the circular frequency, k the wavenumber, g the acceleration of gravity and $A_t = (1-r)/(1+r)$ is the Atwood number with r being the ratio of densities. If the two fluid layers are contained in a closed tank, the two fluid domains must be perfectly symmetric with respect to the interface so that the n^{th} eigenfrequency $\omega_n(r)$ varies as

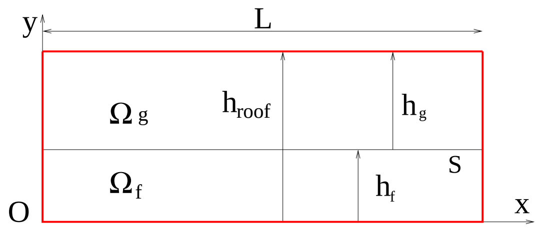
$$\omega_n(r) = \sqrt{\frac{1-r}{1+r}}\omega_n(0) \tag{1}$$

When the geometry is rectangular (with length L and liquid depth h), then $\omega_n^2(0) = \frac{n\pi g}{L}\tanh\frac{n\pi h}{L}$, the proof for (1) is straigtforward. For an arbitrary domain with the interface as a plane of symmetry, it is easy to show that both the velocity potentials and their normal derivative along the interface are equal but opposite in sign; then the identity (1) follows.

In previous works it has been shown that Method of Fundamental Solutions (also known as Desingularized Methods) is effective in simulating nonlinear free surface motions (see Mrabet *et al*, 2014). The main reasons are this method does not require regridding or smoothing during the time marching scheme. Conformal mapping is an essential aspect of the present technique. As a matter of fact, by turning the physical computational domain into a simpler domain, as a half space for example, the impermeability condition on the tank walls is easily accounted for. To this end the following conformal mappings

$$w = -\cos\frac{\pi z}{L}, \qquad w = \cos\frac{\pi}{L}(z - ih_{roof}) \tag{2}$$

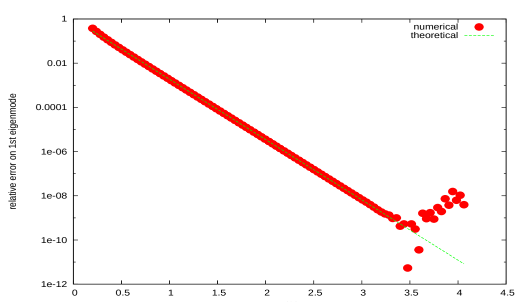
are used for a rectangular tank as described below

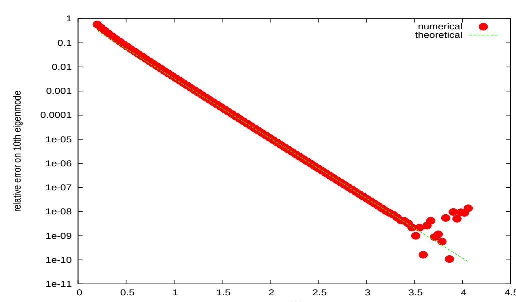


These two independent transformations "flatten" the two vertical walls of the rectangular tank. In practice, the velocity potentials attached to each phase are written as a finite summation of Rankine sources placed at some distance from the actual interface. The number of sources is denoted N and $\delta L/(N-1)$ is the desingularizing distance. When the density ratio is zero, it is shown that the relative error on the n^{th} eigenfrequency varies like

$$E_n = A_n e^{-\pi\delta L\frac{N-n}{N-1}} \tag{3}$$

The same result holds when the density ratio is non zero but the analysis of convergence follows from numerical analysis. The figures below illustrate that result for $r = 0.5$. It shows the decreasing error as a function of δ for the first and tenth modes. On these curves the theoretical error follows from (3).

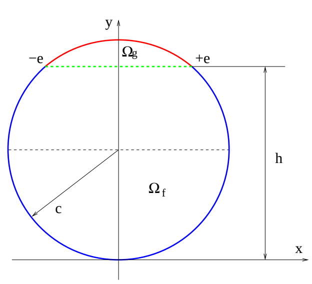




Christiansen (1976) and Pozrikidis (2000) arrived at the same conclusion: the farther the singularities, the better the convergence. However that (non intuitive) result is rather paradoxical. As a matter of fact, when using the same numerical technique but for solving the nonlinear free surface equations, it is shown that the desingularizing distance must be adapted to a small enough value in order to fulfill the criteria of mass and energy conservation (see Scolan, 2010).

2) Eigenfrequencies in a circular tank

We consider a circular tank with radius c filled with two fluids which are not miscible, say Ω_g and Ω_f the corresponding fluid domains as shown in figure below.



The indices f and g refer to the liquid and gas respectively. The density ratio of the two fluids is $r = \rho_g/\rho_f$. The filling height is h measured from the bottom (south pole). The dimension of the interface is defined by $e = \sqrt{c^2 - (h-c)^2}$. The analysis by McIver (1989) provides us with the conformal mapping $x + iy = ih - e\tanh(\zeta/2)$ with $\zeta = \alpha + i\beta$, which turns the physical domain (inner circle) into an infinite strip $\beta \in [-\pi, +\pi]$ and $\alpha \in]-\infty, +\infty[$. The surface of the circle is described by $\beta = C^{ste}$: $\beta = \beta_f$ (such that $\cos\beta_f = \frac{c-h}{c}$) is the image of the circular arc bounding Ω_f and $\beta = \beta_g$ (such that $\cos\beta_g = \frac{h-c}{c}$) is the image of the circular arc bounding Ω_g. It is worth noting that the two angles β_f and β_g are not independent, they are opposite in sign and verify the identity $\beta_f - \beta_g = \pi$.

Following McIver (1989), the general solutions which verify both Laplace equation and impermeability conditions on the wall, read

$$\phi_j(\alpha, \beta, \beta_j) = \int_0^\infty A_j(\tau)\cosh\tau(\beta - \beta_j)\left\{\begin{array}{c}\sin\tau\alpha \\ \cos\tau\alpha\end{array}\right\} d\tau, \qquad j = f \; or \; g \tag{4}$$

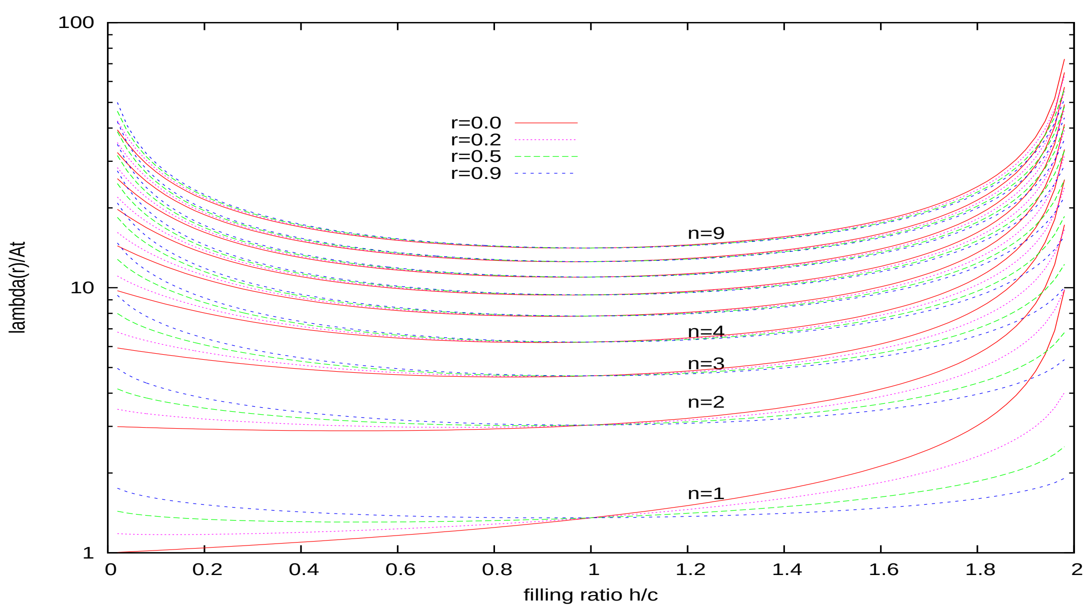
where cos and sin yield the symmetric and antisymmetric modes respectively. These solutions are such that the pressure and the normal velicity at the interface are continuous, yielding the following integral equation for the function B,

$$(1-r)B(\tau') = \lambda\int_0^\infty B(\tau)K(\tau,\tau')d\tau, \qquad \tau' \geq 0, \qquad \lambda = \frac{e\omega^2}{g} \tag{5}$$

$$B(\tau) = A_f(\tau)\sqrt{\tau f(\tau)\sinh\tau\beta_f\cosh\tau\beta_f}, \qquad f(\tau) = 1 - r\frac{\tanh\tau\beta_f}{\tanh\tau\beta_g} \tag{6}$$

$$K(\tau,\tau') = \left[\frac{\tau-\tau'}{\sinh(\tau-\tau')\pi} \mp \frac{\tau+\tau'}{\sinh(\tau+\tau')\pi}\right]\sqrt{\frac{f(\tau)f(\tau')}{\tau\tau'\tanh\tau\beta_f\tanh\tau'\beta_f}} \tag{7}$$

where $\pm$ depends on sin / cos in equation (4), namely the antisymmetric or symmetric modes. Numerically, the integral in (5) is discretised by using a Gauss quadrature formula. To this end, Legendre type sounds better than Laguerre type; after the appropriate change of variable has been done. The next figure shows the variation of the eigenfrequency $\lambda(r)/A_t$ with the filling ratio h/c for the first nine modes. The density ratio varies in the range $(0, 0.2, 0.5, 0.9)$.



When the tank is half filled, equation (1) is perfectly valid. Otherwise, that equation is a rough approximation. It is worth noting that for low filling ratio the frequencies increase with increasing density ratio. This variation is reversed at high filling ratio.

4) Conclusions

The sloshing modes in a tank are rarely computed by accounting for the two fluid system *i.e.* non miscible fluids in a closed tank (see La Rocca *et al*, 2002, 2005). This problem is addressed here by using different approaches. For a rectangular tank, some numerical aspects and convergence criteria of the desingularized technique are detailed. For a circular tank, the quasi-analytical approach proposed by McIver is extended to a two layer system.

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