

# Extended Mild-Slope Equations for Compressible Fluids

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## 1 Introduction

In this paper we derive new forms of the mild-slope equation (MSE) for water waves in a weakly compressible fluid on a slowly varying bathymetry, with surface and bottom disturbances. The MSE is a powerful tool to model the refraction-diffraction dynamics of water waves propagating on a variable bathymetry [1]. Traditionally, mild-slope models are derived by assuming that the wave steepness is small, the fluid is inviscid and incompressible and the motion is irrotational. Furthermore, no disturbances are normally considered both on the free surface and at the bottom of the fluid domain [2]. In this paper we shall find new expressions of the MSE by relaxing the incompressibility hypothesis and considering both surface and bottom disturbances. We shall name the set of new formulae as the extended acoustic-gravity mild-slope equations (EAG-MSE). Such a system of equations can be implemented in numerical models for the early detection of coastal flooding based on the hydro-acoustic precursors of surface gravity waves (see [3]–[5]).

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## 2 Mathematical model

Let us consider the motion of a slightly compressible fluid on a variable bathymetry with surface and bottom disturbances. The wave field is described by the wave equation

$$\nabla^2 \Phi + \Phi_{zz} = \frac{1}{c^2} \Phi_{tt} \quad (1)$$

in the fluid domain, together with the kinematic-dynamic boundary condition on the free surface (see [5])

$$\Phi_{tt} + g\Phi_z = -\frac{P(x, y, t)}{\rho_0}, \quad z = 0, \quad (2)$$

and the no-flux condition at the bottom

$$\Phi_z + \nabla h \cdot \nabla \Phi = -h_t, \quad z = -h(x, y, t), \quad (3)$$

as in [1]. Here,  $\Phi(x, y, z, t)$  is the velocity potential,  $\nabla$  is the horizontal gradient,  $P(x, y, t)$  is a prescribed surface pressure disturbance,  $h(x, y, t)$  denotes the seafloor,  $g$  is the acceleration due to gravity,  $\rho_0$  is the density of the undisturbed fluid,  $c$  is the (assumed constant) speed of sound in water. A reference system  $O(x, y, z)$  is set such that  $(x, y)$  lie on the horizontal plane and  $z$  is the vertical coordinate originating from the undisturbed water plane, positive upwards;  $t$  is time. Physically,  $P$  represents the action of an atmospheric pressure front, which is responsible for the generation of storm surges, while  $h(x, y, t)$  is a seafloor deformation which generates tsunamis [1]. Initially we shall retain both

disturbances and derive a generalised form of the EAG-MSE. Then we shall find particularised expressions of our novel equation for each disturbance. Following [4], let us expand the velocity potential in a Galerkin series

$$\Phi(x, y, z, t) = \sum_{n=0}^{\infty} \psi_n(x, y, t) f_n(z, h(x, y, t)), \quad (4)$$

where the  $\psi_n(x, y, t)$  are unknown functions and the  $f_n(z, h)$  are the solutions of the homogeneous boundary-value problem with the local and instantaneous water depth  $h(x, y, t)$  (see [1]):

$$f_n(z, h(x, y, t)) = \frac{\cosh \beta_n(z + h)}{\cosh \beta_n h}. \quad (5)$$

In the latter, the  $\beta_n = \beta_n(h(x, y, t))$  solve

$$\beta_0 : \quad \omega^2 = g\beta_0 \tanh \beta_0 h \quad (6)$$

$$\beta_n = i\bar{\beta}_n : \quad \omega^2 = -g\bar{\beta}_n \tan \bar{\beta}_n h, \quad (7)$$

where the mathematical problem is formulated for a generic single frequency  $\omega$  of the forcing spectrum [4]. The solution for the complete spectrum is easily obtained by simple Fourier superimposition of the single spectral components ([4], [1]). Physically,  $n = 0$  represents the propagating gravity mode, while  $n > 0$  represents propagating and evanescent hydro-acoustic modes [3]. Let us now introduce the inner product

$$\begin{aligned} & \langle f(x, y, z, t), g(x, y, z, t) \rangle \\ &= \int_{-h}^0 f(x, y, z, t) g(x, y, z, t) dz, \end{aligned} \quad (8)$$

so that  $\langle f_n, f_m \rangle = 0$  if  $n \neq m$ . We shall also expand the surface and bottom disturbances accordingly, i.e.

$$P(x, y, t) = \sum_{n=0}^{\infty} P_n(x, y, t) f_n(x, y, z, t), \quad (9)$$

$$h(x, y, t) = \sum_{n=0}^{\infty} h_n(x, y, t) f_n(x, y, z, t) \quad (10)$$

respectively, where

$$P_n(x, y, t) = \frac{\langle P, f_n \rangle}{\langle f_n, f_n \rangle} \quad (11)$$

and

$$h_n(x, y, t) = \frac{\langle h, f_n \rangle}{\langle f_n, f_n \rangle}. \quad (12)$$

Following [2] (Section 7.2.3.3), first calculate

$$\left\langle f_m, \nabla^2 \Phi + \Phi_{zz} - \frac{1}{c^2} \Phi_{tt} \right\rangle = 0, \quad (13)$$

from Eq.(1), with  $m = 0, 1, \dots$ . Then substitute the expansion (4) into (13) to get

$$\begin{aligned} & \sum_{n=0}^{\infty} [\nabla^2 \psi_n \langle f_m, f_n \rangle + 2\nabla \psi_n \cdot \langle f_m, \nabla f_n \rangle \\ & + \psi_n \langle f_m, \nabla^2 f_n \rangle + \psi_n \langle f_m, f_{nzz} \rangle \\ & - \frac{1}{c^2} (\psi_{n_{tt}} \langle f_m, f_n \rangle + 2\psi_{n_t} \langle f_m, f_{n_t} \rangle \\ & + \psi_n \langle f_m, f_{n_{tt}} \rangle)] = 0, \end{aligned} \quad (14)$$

for the governing equation. Similarly, substituting (4) into (2) and (3), one gets

$$\begin{aligned} & \sum_{n=0}^{\infty} \psi_n f_m f_{n_z} = -\frac{1}{g} \sum_{n=0}^{\infty} (\psi_{n_{tt}} f_m f_n \\ & + 2\psi_{n_t} f_m f_{n_t} + \psi_n f_m f_{n_{tt}}) \\ & - \frac{1}{\rho_0 g} \sum_{n=0}^{\infty} (P_{n_t} f_m f_n + P_n f_m f_{n_t}), \quad z = 0 \end{aligned} \quad (15)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \psi_n f_m f_{n_z} = -\sum_{n=0}^{\infty} (\psi_n f_m \nabla f_n \cdot \nabla h \\ & + \nabla \psi_n \cdot \nabla h f_m f_n + h_{n_t} f_m f_n + \\ & + h_n f_m f_{n_t}), \quad z = -h \end{aligned} \quad (16)$$

for the boundary conditions, respectively. Further application of the Green integral formula

$$\langle f_m, f_{n_{zz}} \rangle = -\langle f_{m_z}, f_{n_z} \rangle + [f_m f_{n_z}]_{z=-h}^{z=0} \quad (17)$$

and usage of (15)–(16), together with the properties

$$f_{n_t} = f_{n_h} h_t, \quad f_{n_{tt}} = f_{n_{hh}} (h_t)^2 + f_{n_h} h_{tt}$$

and

$$(f_{n_h})_{z=0} = (f_{n_{hh}})_{z=0} = 0,$$

transforms (14) into the sought system of expressions:

$$\sum_{n=0}^{+\infty} [a_{mn} \nabla^2 \psi_n + \mathbf{b}_{nm} \cdot \nabla \psi_n$$

$$\begin{aligned}
& + \left( c_{mn} - \frac{\omega^2}{g} \right) \psi_n - \frac{1}{g} \psi_{ntt} \Big] \\
& = \frac{1}{c^2} \sum_{n=0}^{\infty} (a_{mn} \psi_{ntt} + d_{mn} \psi_{nt} \\
& + e_{mn} \psi_n) + \frac{1}{\rho_0 g} \sum_{n=0}^{+\infty} P_{nt} \\
& - \sum_{n=0}^{+\infty} [g_{mn} h_{nt} + l_{mn} h_n], \quad m \in \mathbb{N}, \quad (18)
\end{aligned}$$

after lengthy algebraic simplifications. In Eq. (18),

$$\begin{aligned}
a_{mn} &= \langle f_m, f_n \rangle, \\
\mathbf{b}_{mn} &= 2 \langle f_m, \nabla f_n \rangle + (f_m f_n)_{z=-h} \nabla h, \\
c_{mn} &= - \langle f_{mz}, f_{nz} \rangle + \langle f_m, \nabla^2 f_n \rangle \\
& + (f_m \nabla f_n)_{z=-h} \cdot \nabla h + \frac{\omega^2}{g}, \\
d_{mn} &= 2 \langle f_m, f_{n_h} \rangle h_t, \\
e_{mn} &= \langle f_m, f_{n_{hh}} \rangle h_t^2 + \langle f_m, f_{n_h} \rangle h_{tt}, \\
g_{mn} &= (f_m f_n)_{z=-h} \\
l_{mn} &= (f_m f_{n_h})_{z=-h} h_t,
\end{aligned}$$

All the above terms depend on  $x, y, t$  via the  $f_n$  and  $h$  (see Eq.5). Expression (18) is the novel EAG-MSE for waves in a weakly compressible fluid generated by surface pressure disturbances and seafloor deformations. It represents both gravity ( $n = 0$ ) and hydro-acoustic ( $n > 0$ ) waves of given frequency  $\omega$  [4, 5]. Note that in the limit  $c \rightarrow \infty$ , assuming  $P = h_t = 0$  and steady-state harmonic oscillations of frequency  $\omega$ , (18) fully corresponds to Massel's MSE (see equation 16 in [2]).

### 3 The adiabatic approximation

The adiabatic hypothesis is a frequent approximation undertaken in the modelling of acoustic-gravity waves. Within such an approximation, one neglects the cross-coupling terms in the governing equations, not allowing the normal modes to interact among themselves. This framework allows for much quicker computations, usually without significant loss of accuracy [6]. Assuming that each mode propagates without interacting with the others, the EAG-MSE (18) simpli-

fies into

$$\begin{aligned}
& a_{nn} \nabla^2 \psi_n + \mathbf{b}_{nn} \cdot \nabla \psi_n + (c_{nn} - \omega^2/g) \psi_n \\
& - \frac{1}{g} \psi_{ntt} = \frac{1}{c^2} (a_{nn} \psi_{ntt} + d_{nn} \psi_{nt} + e_{nn} \psi_n) \\
& + \frac{1}{\rho_0 g} P_{nt} - g_{nn} h_{nt} - l_{nn} h_n, \quad n = 0, 1, \dots \quad (19)
\end{aligned}$$

which we name the adiabatic acoustic-gravity mild-slope equation (AAG-MSE).

#### 3.1 AAG-MSE for tsunamis (Sammarco *et al.*'s MSEWC)

Let us consider the case  $P = 0$ ,  $h = h(x, y, t)$ . For a geophysical flow over a large area with a slowly varying bottom, such as a tsunami, the dependence of  $f_n$  on  $h$  can be safely neglected, so that  $f_n = f_n(z)$  (see [4]) and

$$d_{nn} = e_{nn} = l_{nn} = 0. \quad (20)$$

Furthermore, for a slowly varying depth, the higher-order terms

$$\nabla^2 f_n = O(\nabla^2 h), \quad \nabla f_n \cdot \nabla h = O(|\nabla h|^2),$$

can be neglected too, so that

$$c_{nn} = \beta_n^2 a_{nn}. \quad (21)$$

Finally note that

$$a_{nn} \nabla^2 \psi_n = \nabla \cdot (a_{nn} \nabla \psi_n) - \nabla a_{nn} \cdot \nabla \psi_n, \quad (22)$$

where

$$\nabla a_{nn} = a_{nn_h} \nabla h = \frac{\partial}{\partial h} \int_{-h}^0 f_n^2 dz \nabla h = \mathbf{b}_{nn} \quad (23)$$

as an application of the Leibniz integral rule. Substituting (20)–(23) into (19), the latter yields finally

$$\begin{aligned}
& \psi_{ntt} \left( \frac{C_n}{c^2} + \frac{1}{g} \right) - \nabla \cdot (C_n \nabla \psi_n) \\
& + \left( \frac{\omega^2}{g} - \beta_n^2 C_n \right) \psi_n = D_n h_t. \quad (24)
\end{aligned}$$

In Eq. (24),

$$C_n = a_{nn} = \frac{2\beta_n h + \sinh(2\beta_n h)}{4\beta_n \cosh^2(\beta_n h)}, \quad (25)$$

$$D_n = \frac{\langle g_{nn}, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{4 \tanh(\beta_n h)}{2\beta_n h + \sinh(2\beta_n h)}. \quad (26)$$

Expression (24) is the MSE for weakly compressible fluids (MSEWC) found by Sammarco *et al.* [4]. The latter allows to model the hydro-acoustic waves travelling fast ahead of an incoming tsunami generated by a seafloor movement. Such hydro-acoustic waves leave a distinctive signature on the bottom pressure that can be used for the early detection of tsunamis [3, 4].

### 3.2 AAG-MSE for storm surges

We shall now consider the case of a fixed bathymetry  $h = h(x, y)$  and a surface pressure distribution  $P(x, t)$ . In such a case,  $d_{nn} = e_{nn} = l_{nn} = 0$ . Therefore, using again (23), expression (19) simplifies to

$$\psi_{n_{tt}} \left( \frac{C_n}{c^2} + \frac{1}{g} \right) - \nabla \cdot (C_n \nabla \psi_n) + \left( \frac{\omega^2}{g} - \beta_n^2 C_n \right) \psi_n = -\frac{B_n}{\rho_0 g} P_t, \quad (27)$$

where

$$B_n = \frac{2 \sinh(2\beta_n h)}{2\beta_n h + \sinh(2\beta_n h)}. \quad (28)$$

Expression (27) allows to model the hydro-acoustic precursors of storm waves over a variable 3D bathymetry.

## 4 Numerical computations

Numerical computations of the MSEWC have already been performed in [4], where equation (24) has been solved for constant and mild-slope domain configurations. Numerical analysis of the AAG-MSE for storm waves (27) is currently being performed for a travelling pressure perturbation. Preliminary results will be presented at the workshop.

The numerical model solves the partial differential equation by means of the Finite Element Method. The three dimensional domain is discretised into tetrahedral elements, whose minimum element size is set to be at least 1/10 of the simulated wave length, in order to correctly reproduce the wave field. Neumann type boundary conditions are applied at the boundaries of the domain. Since the mathematical problem is hyperbolic, it is solved by means of a time-marching numerical scheme. For the

efficiency of the time-stepping algorithm, it is important to assemble the time independent matrices only once. We use the generalised- $\alpha$  method, which is a one-step implicit method for solving the transient problem. Frequency bands of defined width are selected to discretise the forcing spectrum and to solve a set of equations, as (27), each one calculated using the carrier frequency of the selected band. Then a broad frequency spectrum can be simulated by superimposing the results. The numerical solution is obtained using the software COMSOL Multiphysics.

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