A new linearization method for vectorial Morison equation

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Introduction

For offshore and naval applications, the wave forces acting on slender parts of the bodies can be described by the Morison equation, which expresses the overall force in terms of a nonlinear drag force and a linear inertia force that define an additional damping and added mass.

In frequency domain, the drag term of the Morison force needs to be linearized. In case of irregular waves, Borgman [1] linearized the wave drag force in the Morison Equation using the random Gaussian process assumption for the wave velocity. This linearized form was developed for a fixed vertical cylinder under the action of a unidirectional wave.

However, for seakeeping and mooring analysis, the vectorial Morison equation is used to evaluate the wave force on slender bodies in multi-directional flows. Therefore, this linearized form needs to be modified since it does not obey vector operation rules.

The purpose of this paper is to present a new linearization method for the multi-directional flow. A comparison between the new and the classical linearization will be presented against the time domain simulation in the case of cylinder.

Morison equation and limitations of the existing linearization

In unidirectional wave flow, the Morison load on a fixed circular cylinder (per unit length), without current, is given by:

$$f_{Morison} = (1 + C_m)\rho S\dot{u} + \frac{1}{2}C_d\rho D\left|u\right|u \tag{1}$$

Where D stands for the cylinder diameter and S its section, C_d the drag coefficient, C_m the inertia coefficient and u the wave field velocity projected on the perpendicular plane to the principal axis element. If the cylinder is in motion x, the velocity is replaced by the relative-velocity projected on the same plane:

$$f_{Morison} = (1 + C_m)\rho S\dot{u} - C_m\rho S\ddot{x} + \frac{1}{2}C_d\rho D\left|u - \dot{x}\right|(u - \dot{x}) \tag{2}$$

The drag f_d and the inertia f_i loads can be written as:

$$f_i = (1 + C_m)\rho S\dot{u} - C_m\rho S\ddot{x} \tag{3}$$

$$f_d = \frac{1}{2}C_d\rho D\left|u - \dot{x}\right|(u - \dot{x}) \tag{4}$$

In the vectorial case, the Morison equation involves only the two velocity components along the plane normal to the axis of the element. Therefore, for the cylinder local coordinate system (O, x, y, z) such as (Oz) the element axis, the Morison equation can be written as:

$$f_{Morison} = (1 + C_m)\rho S\begin{bmatrix}\dot{u_x}\\ \dot{u_y}\end{bmatrix} - C_m\rho S\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} + \frac{1}{2}C_d\rho D\sqrt{(u_x - \dot{x})^2 + (u_y - \dot{y})^2}\begin{bmatrix}u_x - \dot{x}\\ u_y - \dot{y}\end{bmatrix} \tag{5}$$

In irregular waves and without current, u is a zero mean Gaussian random process. With the same assumptions as above, Borgman showed that the autocorrelation function $R_{f_d f_d}$ of the drag force, can be written simply in terms of the wave velocity autocorrelation function R_{uu} ([2] and [1]):

$$R_{f_d f_d}(\tau) = (\frac{1}{2}C_d\rho D)^2 R_{uu}(0)^2 G(r) \tag{6}$$

Where:

$$r = \frac{R_{uu}(\tau)}{R_{uu}(0)} \tag{7}$$

$$G(r) = \frac{1}{\pi}\left((4r^2 + 2)arsin((r) + 6r\sqrt{1 - r^2}\right) \tag{8}$$

With the first order linearization of the function $G(r)$ (because $0 \leqslant r \leqslant 1$), it is possible to write:

$$R_{f_d f_d}(\tau) \approx (\frac{1}{2} C_d \rho D)^2 R_{uu}(0)^2 \frac{8}{\pi} r \tag{9}$$

The Energy spectral density is then calculated, using the Fourier transform of the above equation:

$$S_{f_d f_d}(f) \approx (\frac{1}{2} C_d \rho D)^2 \frac{8}{\pi} \sigma_u^2 S_{uu}(f) \tag{10}$$

Finally, the linearized force is given by:

$$f_d \approx \frac{1}{2} \rho C_d D \sqrt{\frac{8}{\pi}} \sigma_u u \tag{11}$$

Where σ_u is the standard deviation of the wave velocity. In the case where the cylinder is in motion, we suppose that the relative-velocity is also a zero mean random Gaussian process so we can write:

$$f_d \approx \frac{1}{2} C_d \rho D \sqrt{\frac{8}{\pi}} \sigma_{u-\dot{x}} (u - \dot{x}) \tag{12}$$

In the vectorial case, this method faces two problems:

1. A common linearization constant cannot be calculated for the two components of the Morison drag force.
2. The characteristics of water particle velocity (orbital velocity for example) cannot be modeled efficiently without considering the correlation between between the 2 directional velocities.

One way to solve the first problem is to linearize each component of the drag force separately by considering a unidirectional flow following each wave velocity direction:

$$\overrightarrow{f_d} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \frac{1}{2} C_d \rho D \sqrt{\frac{8}{\pi}} \begin{bmatrix} \sigma_{u_x} & 0 \\ 0 & \sigma_{u_y} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \tag{13}$$

However, this method will underestimate the drag force since in its original form (Eq.5), the velocity norm depends on the two other velocity components and will be always higher than the norm of one component. Thus, a more complete model needs to be developed for an accurate linearization form.

Analytical model

We consider a fixed cylinder in 3D random wave flow. The cylinder is not supposed to be necessarily vertical since we will work on its local coordinate system $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{n})$ with $\overrightarrow{n}$ the cylinder's vector axis. We will be limited in our study to a two-dimensional case since the Morison equation depend on the projected wave velocity as mentioned previously. Thus, the drag force and the wave velocity can be written in the local coordinate system $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{n})$ as:

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad , \quad \overrightarrow{f_d} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{14}$$

Extending the Borgman's linearization to a two-dimensional case seems to be difficult for the reason that the autocorrelation function of each force component is complicated to determine analytically (need to calculate multiple integrals of 4 variables: $u_1(t)$, $u_2(t)$, $u_1(t+\tau)$ and $u_2(t+\tau)$). For simplification, a classic random Gaussian vector model will be used for $\overrightarrow{u}$ so its probability density function can be defined as:

$$p(u_1, u_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-{\rho_{12}}^2}} exp\left(-\frac{\frac{{u_1}^2}{{\sigma_1}^2} + \frac{{u_1}^2}{{\sigma_1}^2} - \frac{2u_1u_2\rho_{12}}{\sigma_1\sigma_2}}{2(1-\rho_{12}^2)}\right) \tag{15}$$

Where σ_1 and σ_2 are respectively the standard deviations of the two zero mean random Gaussian variables u_1 and u_2 and $\rho_{12} = \frac{cov(u_1,u_2)}{\sigma_1\sigma_2}$ the correlation coefficient. The idea of this linearization is based on the energy dissipation of the Morison drag force which will be calculated for the exact and the linearized formulations. The linearization coefficient must provide the same energy dissipation for each model.

Mathematically, we define the energy dissipation as the expected value $< . >$ of the drag force mechanical power $\overrightarrow{f_d}.\overrightarrow{u}$. For the linear model, we can write:

$$\overrightarrow{f_L} = \frac{1}{2} C_d \rho D K \overrightarrow{u} = K_D K \overrightarrow{u} \tag{16}$$

So the dissipated energy E_L is calculated by:

$$E_L = < \overrightarrow{f_L}.\overrightarrow{u} > = K_D K < u_1^2 + u_2^2 > = K_D K \left(\sigma_1^2 + \sigma_2^2\right) \tag{17}$$

For the exact model, we can write:

$$E = < \overrightarrow{f_d}.\overrightarrow{u} > = K_D < \left(u_1^2 + u_2^2\right) \|\overrightarrow{u}\| > = K_D < \left(u_1^2 + u_2^2\right)^{\frac{3}{2}} > \tag{18}$$

Using the expected value mathematical definition:

$$E = \frac{K_D}{2\pi\sigma_1\sigma_2\sqrt{1-{\rho_{12}}^2}} \iint\limits_{\mathbb{R}^2} \left(u_1^2 + u_2^2\right)^{\frac{3}{2}} exp\left(-\frac{\frac{{u_1}^2}{{\sigma_1}^2} + \frac{{u_1}^2}{{\sigma_1}^2} - \frac{2u_1u_2\rho_{12}}{\sigma_1\sigma_2}}{2(1-\rho_{12}^2)}\right) \mathrm{d}u_1 \mathrm{d}u_2 \tag{19}$$

With the polar transformation: $u_1 = rcos(\theta)\sigma_1$ and $u_2 = rsin(\theta)\sigma_2$, the previous double integral can be simplified in a single elliptic integral:

$$E = \frac{3K_D}{2\sqrt{2\pi}} \int_0^{2\pi} \frac{\left((\sigma_1 cos(\theta))^2 + (\sigma_2 sin(\theta))^2\right)^{\frac{3}{2}}}{\sqrt{1-\rho_{12} sin(2\theta)}} \mathrm{d}\theta \tag{20}$$

By considering $E = E_L$, we obtain:

$$K = \frac{3}{2\sqrt{2\pi}} \int_0^{2\pi} \frac{\left((acos(\theta))^2 + ((1-a)sin(\theta))^2\right)^{\frac{3}{2}}}{\sqrt{1-\rho_{12} sin(2\theta)}} \mathrm{d}\theta \ \sqrt{\sigma_1^2 + \sigma_2^2} \tag{21}$$

Where:

$$a = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \tag{22}$$

Finally, the linearized force is given by:

$$\overrightarrow{f_L} = \frac{3K_D}{2\sqrt{2\pi}} \int_0^{2\pi} \frac{\left((acos(\theta))^2 + ((1-a)sin(\theta))^2\right)^{\frac{3}{2}}}{\sqrt{1-\rho_{12} sin(2\theta)}} \mathrm{d}\theta \ \sqrt{\sigma_1^2 + \sigma_2^2} \ \overrightarrow{u} \tag{23}$$

As the result shows, the linearization coefficient is expressed as a function of two non-dimensional parameters. Using this two-parameter model, we fulfill requirements imposed by the two problems mentioned in the previous section:

- The parameter a represents the two-dimensional aspect of the flow (problem 1).
- The parameter ρ_{12} expresses the correlation relation between the two velocity components u_1 and u_2 and therefore the wave field characteristics: linear, orbital, ... (problem 2).

For verification, in the case of a unidirectional wave field following $\overrightarrow{e_1}$ direction ($a = 1$ and $\rho_{12} = 0$), we have:

$$K = \frac{3}{2\sqrt{2\pi}} \int_0^{2\pi} |cos(\theta)|^3 \mathrm{d}\theta \ \sigma_1 = \sqrt{\frac{8}{\pi}} \sigma_1 \tag{24}$$

We obtain the Borgman linearization constant in the unidirectional case.

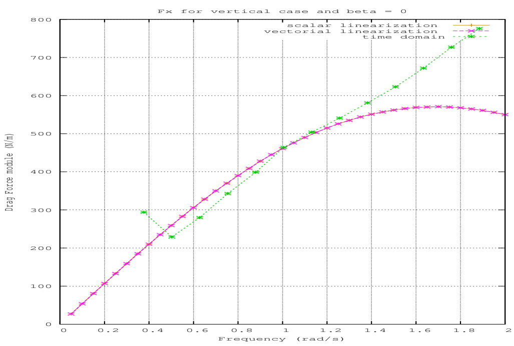
For a 3D wave field, with a structure containing several Morison elements, each element is discretized on Gauss integration points. The relative velocity of each integration point is projected in the local coordinate system of the element. Next, K and $\overrightarrow{f_L}$ are calculated in the local coordinate system then expressed in the global system coordinate and summed.

Simulation results and discussion

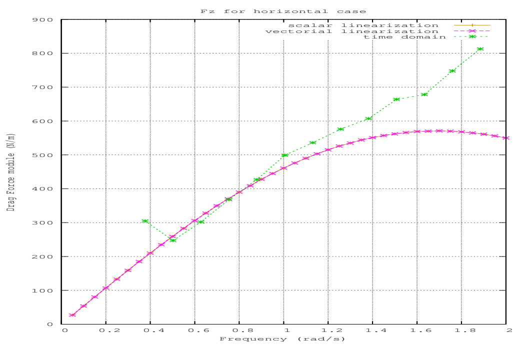
In order to test this linearization, the Morison drag force is calculated in irregular waves for a fixed cylinder in time and frequency domains. Cylinder dimensions are ($R = 1m$ x $H = 3m$) and $C_d = 0.7$. As a wave model, a JONSWAP spectrum is used with the parameters $H_s = 1.0m$, $T_p = 12.0s$ and $\gamma = 1.0$. The water depth is infinite. The wave time signal has been generated for 3 hours using a linear reconstruction of the wave spectrum with a random phase for each frequency component. The power spectral density (PSD) of each signal has been calculated in order to obtain the drag force RAOs.

To show the difference between the two linearizations (scalar and vectorial) in different configurations, the force is calculated for the vertical and the horizontal position of the cylinder (linear and orbital wave fields) for two headings $\beta = 0°$ (head) and $\beta = 45°$ (diagonal). The axis of cylinder is (Oz) in the vertical case and (Ox) in the horizontal case. The wave heading is defined by the angle between the propagation direction and the positive direction of the axis (Ox).

The figures bellow give the RAOs of the drag force components. As the results show, the frequency domain solution seems to underestimate the drag force, due to the linearization effect. In addition, for $\beta = 0°$, the projected wave field is unidirectional so the two linearizations give the same results (figure 1 for the horizontal and vertical case and are in agreement with the time domain solution (green plot).



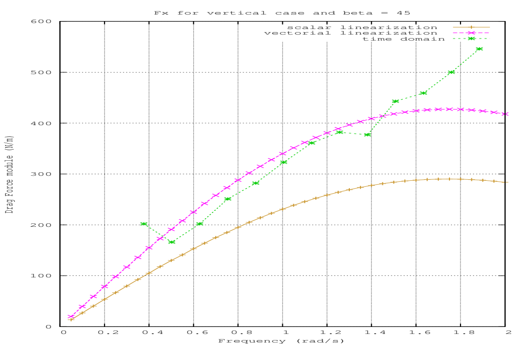
(a) F_x Vertical case



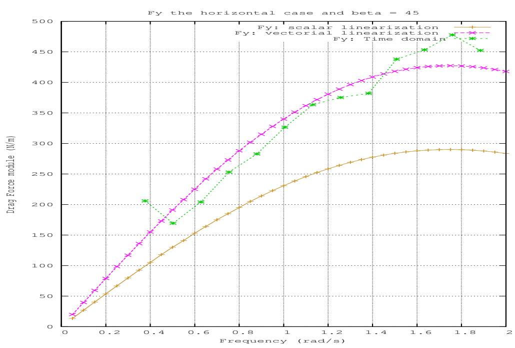
(b) F_z Horizontal case

Figure 1: $\beta = 0°$

However, for the diagonal heading $\beta = 45°$ (figure 2), since the drag force will depend on more than one velocity direction ((Ox) and (Oy) in the vertical case and (Oz) and (Oy) in the horizontal case), the vectorial linearization (pink plot) is more accurate and the scalar linearization (brown plot) underestimates the drag force especially for the horizontal case which corresponds to the circular wave filed.



(a) F_x Vertical case



(b) F_y Horizontal case

Figure 2: $\beta = 45°$

References

[1] Leon E. Borgman. Random hydrodynamic forces on objects. *Ann. Math. Statist.*, pages 37–51, 1967.

[2] R. L. Wiegel. Waves and their effects on pile-supported structures. pages 1–10, 1969.