Scattering by rings of vertical cylinders

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Highlights:

• Foldy-type methods are developed to study the title problem with equally spaced cylinders around a ring.

• Explicit solutions are obtained by exploiting the circulant structure implied by the special geometry.

1. Introduction

It is 25 years since the publication of "Linton & Evans" [1] on "The interaction of waves with arrays of vertical circular cylinders"; it is their most cited joint work. The paper describes an exact method in which separation of variables and addition theorems are combined, leading to an infinite system of linear algebraic equations. The method itself is much older, and goes back to a paper by Záviška from 1913. It has been used in many contexts, and extended in many directions; for discussion and many references, see [2, Chapter 4]. Apart from using the basic method for water-wave problems, Linton & Evans [1] also showed that the computation of the pressure near, or on, any one cylinder could be simplified considerably.

In this paper, we are interested in the scattering of an incident plane wave by N identical vertical circular cylinders arranged in a particular way: in a horizontal plane (plan view), there are N circles (radius a) with their centres located on, and equally spaced around, a larger circle (radius b). We call this geometrical configuration a *ring* or a *cage*.

Of course, the Linton–Evans method can be, and has been, applied to scattering by a ring of cylinders. See, for example, [3, 4, 5]. However, we are especially interested when N is large, so that we have many small circles around the ring with small gaps between them.

Intuitively, we expect that, in the limit (when there are no gaps), we should approach the solution for scattering by a single large cylinder (with cross-section of radius b). Can this be shown, and, if so, how fast is the limit achieved?

The problem we have described is reminiscent of a problem in electrostatics, a *Faraday cage*. Thus, a metal enclosure protects its inhabitants from external electrical discharges, as first demonstrated by Michael Faraday in 1836. If the metal has small holes or gaps, protection is no longer perfect.

In a recent paper [6], we gave an analysis of such problems, for both electrostatics (Laplace's equation) and acoustics (Helmholtz equation); the latter is of most relevance here. The cylinders comprising the wires in the cage were assumed to be small, both geometrically ($a \ll b$) and acoustically ($ka \ll 1$, where $2\pi/k$ is the incident wavelength). For the scattering itself, we used a much simplified version of Linton–Evans, one in which the scattering by each circular cylinder is represented by a single term (proportional to $H_0(kr)$, see below) instead of the usual infinite separation-of-variables series. This leads to *Foldy's method* [7], [2, §8.3], which takes account of all the multiple scattering effects. The result is an $N \times N$ linear algebraic system. This reduction works for N scatterers at more-or-less arbitrary locations. However, for a ring of equally-spaced identical scatterers, the matrix occurring has a special structure: it is a *circulant* matrix. This means that it can be inverted explicitly, using a discrete Fourier transform, and then the behaviour of the solution as N grows can be analysed. It turns out that the expected limit is achieved but the limit is approached slowly, as $N^{-1}\log N$.

So far, we have not mentioned the boundary condition on each cylinder. The exact Linton–Evans approach can accommodate any choice, such as a Dirichlet condition (pressure or potential specified, "sound-soft" in acoustics) or a Neumann condition (normal velocity specified, "sound-hard" in acoustics). In the context of water waves, the usual case is the Neumann condition, as imposed in [1].

For the simplified Foldy-type analysis described above, the underlying assumption is that each cylinder scatters *isotropically*: note the presence of $H_0(kr)$ with no dependence on the polar angle. This is entirely appropriate for Dirichlet problems because we know that small ($ka \ll 1$) sound-soft circles really do scatter like a monopole. On the other hand, sound-hard circles do not scatter isotropically: both monopole and dipole contributions are equally important and must be retained. The dipole gives a directional dependence to the waves scattered by one circle, and this must be incorporated into the calculation of the multiply scattered waves when there are N circles.

2. Basic formulation

A plane wave is incident upon N vertical cylinders in water of depth H. As usual, we factor out the depth dependence and write the velocity potential for the scattered waves as

$$\mathrm{Re}\{u(x,y)\cosh k(H-z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$$

where $z=0$ is the free surface, $z=H$ is the flat bottom and k is the positive real solution of $\omega^2 = gk\tanh kH$. Denote the cross-section of the jth cylinder in the xy-plane by $\mathcal{C}_j$; it is centred at $\boldsymbol{r}_j$. Then u satisfies $(\nabla^2 + k^2)u = 0$ outside all the $\mathcal{C}_j$, together with a radiation condition and a boundary condition on each $\mathcal{C}_j$.

3. Basic Foldy approach

Foldy's method starts by assuming isotropic scattering. This means that, near $\mathcal{C}_j$, the scattered field at $\boldsymbol{r}$ is approximated by

$$A_j G(\boldsymbol{r}-\boldsymbol{r}_j),$$

where A_j is an unknown amplitude, $G(\boldsymbol{r}) = H_0(k|\boldsymbol{r}|)$ is the free-space Green's function and $H_n(w) \equiv H_n^{(1)}(w)$ is a Hankel function. The total field is represented as

$$u(\boldsymbol{r}) = u_{\mathrm{inc}}(\boldsymbol{r}) + \sum_{j=1}^{N} A_j G(\boldsymbol{r}-\boldsymbol{r}_j), \tag{1}$$

where we will take $u_{\mathrm{inc}}(\boldsymbol{r}) = u_{\mathrm{inc}}(x,y) = \mathrm{e}^{\mathrm{i}kx}$.

The field incident on $\mathcal{C}_n$ in the presence of all the other scatterers is

$$\begin{aligned} u_n(\boldsymbol{r}) &\equiv u(\boldsymbol{r}) - A_n G(\boldsymbol{r}-\boldsymbol{r}_n) \\ &= u_{\mathrm{inc}}(\boldsymbol{r}) + \sum_{\substack{j=1\\ j\neq n}}^{N} A_j G(\boldsymbol{r}-\boldsymbol{r}_j). \end{aligned} \tag{2}$$

This "incident" field is scattered by $\mathcal{C}_n$. We characterize this process by

$$A_n = g u_n(\boldsymbol{r}_n). \tag{3}$$

This makes the strength of the scattered wave from $\mathcal{C}_n$, A_n, proportional to the field acting on it, $u_n(\boldsymbol{r}_n)$. The parameter g (Foldy's "scattering coefficient") can be chosen as $g = -[J_0(ka)]/[H_0(ka)]$, where J_n is a Bessel function and each $\mathcal{C}_j$ has radius a; if the scatterers were different, we would have written g_n in (3).

Finally, evaluating (2) at $\boldsymbol{r}_n$ gives, after using (3),

$$\frac{1}{g}A_n = u_{\mathrm{inc}}(\boldsymbol{r}_n) + \sum_{\substack{j=1\\ j\neq n}}^{N} A_j\, G(\boldsymbol{r}_n-\boldsymbol{r}_j), \tag{4}$$

for $n = 1, 2, \ldots, N$. This is a linear $N\times N$ system for A_j. Then the total field is given by (1).

4. Application to a ring of soft cylinders

Here, we summarise the results from [6]. There are N small sound-soft circles arranged so that their centres (at $\boldsymbol{r}_j$) are equally spaced around a larger circle of radius b, centred at the origin. Let $h = 2\pi/N$ be the angular spacing between adjacent scatterers. Then, using plane polar coordinates, r and θ, $\boldsymbol{r}_j$ is at $r=b$, $\theta = \theta_j = jh$. The distance between the jth and nth scatterers is

$$|\boldsymbol{r}_n - \boldsymbol{r}_j| = 2b\,|\sin([n-j]\pi/N)|. \tag{5}$$

Then the $N\times N$ Foldy system (4) simplifies to

$$\sum_{j=1}^{N} K_{n-j}A_j = f_n, \quad n = 1, 2, \ldots, N, \tag{6}$$

where $f_n = -u_{\mathrm{inc}}(\boldsymbol{r}_n)$, $K_0 = -g^{-1}$,

$$K_j = H_0(2kb\,|\sin(j\pi/N)|), \quad j \not\equiv 0 \bmod N \tag{7}$$

and K_j is N-periodic: $K_{j+mN} = K_j$, $m = \pm1, \pm2, \ldots$.

Richmond [8] and Wilson [9] gave numerical solutions of (6). Much later, Vescovo [10] noticed that the system matrix in (6) is a circulant matrix, which means that (6) can be solved explicitly using the discrete Fourier transform. Thus, let $\varpi = \mathrm{e}^{2\pi\mathrm{i}/N}$. Multiply (6) by ϖ^{mn}, sum over n and use the N-periodicity of K_n. This gives

$$\tilde{A}_m = \tilde{f}_m/\tilde{K}_m, \tag{8}$$

where

$$\tilde{A}_m = \sum_{j=1}^{N} A_j\varpi^{mj}, \quad A_n = \frac{1}{N}\sum_{j=1}^{N}\tilde{A}_j\varpi^{-nj}, \tag{9}$$

$$\tilde{f}_m = \sum_{j=1}^{N} f_j\varpi^{mj}, \quad \tilde{K}_m = \sum_{j=1}^{N} K_j\varpi^{mj}. \tag{10}$$

Finally, invert the discrete Fourier transform of $\{A_j\}$, $\{\tilde{A}_m\}$, using the second of (9).

Having determined A_n, we can calculate the field everywhere, using (1). In addition, we can investigate analytically what happens as N grows.

Thus, for $u_{\mathrm{inc}} = \mathrm{e}^{\mathrm{i}kx}$, we obtain

$$\tilde{f}_m = -\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}kb\cos jh}\,\mathrm{e}^{\mathrm{i}mjh}.$$

Write this formula (suggestively) as

$$\frac{\tilde{f}_m}{N} = h\sum_{j=1}^{N} f(jh) \quad \text{with} \quad f(\theta) = -\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}kb\cos\theta}\mathrm{e}^{\mathrm{i}m\theta}.$$

We recognise the sum. It is what we would have obtained if we had used the repeated trapezium rule to compute $\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$, noting that f is 2π-periodic. As f is also very smooth, we know that the convergence is exponentially fast. Hence, evaluating the integral gives

$$N^{-1}\tilde{f}_m \sim -\mathrm{i}^m J_m(kb) \quad \text{as } N\to\infty. \tag{11}$$

This rapid convergence is encouraging, but the behaviour of $\tilde{K}_m$ is quite different. From (7) and (10),

$$\tilde{K}_m = -\frac{1}{g} + \sum_{j=1}^{N-1} v(jh), \qquad (12)$$

where

$$v(\theta) = \mathrm{e}^{\mathrm{i}m\theta} H_0(2kb\,|\sin(\theta/2)|)$$

is 2π-periodic but has log singularities at $\theta = 0$ and $\theta = 2\pi$. The sum in (12) looks like the trapezoidal rule in which the endpoint contributions have been "ignored"; the properties of such sums have been analysed by Sidi [11]. Using his results, we find that

$$\frac{1}{N}\tilde{K}_m \sim -\frac{1}{Ng} + \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,\mathrm{d}\theta + \frac{2\mathrm{i}}{\pi N}\log N$$

as $N \to \infty$. The integral can be evaluated. Eventually, after combining with (8) and (11), we find that

$$\tilde{A}_n = -\mathrm{i}^n/H_n(kb) + O(N^{-1}\log N) \quad \text{as } N \to \infty. \qquad (13)$$

The leading approximation can be used to confirm our expectations. For example, the far-field pattern of the ring approaches that for scattering by a sound-soft circle of radius b [2, eqn (4.10)], but this limit is approached very slowly. Similarly, the total field at the origin is $O(N^{-1}\log N)$ as $N \to \infty$.

4. Extended Foldy approach

Rigid (sound-hard) scatterers always induce a dipole field. Foldy's method can be generalized to cover these situations [2, §8.3.3]. Thus, suppose that, near the jth scatterer, the scattered field is given by

$$A_j G(\boldsymbol{r} - \boldsymbol{r}_j) + \boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j), \qquad (14)$$

where A_j is an amplitude, $\boldsymbol{q}_j$ is a vector,

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{1}{k}\,\mathrm{grad}\, G(\boldsymbol{r}) = -\frac{\hat{\boldsymbol{r}}}{k}\frac{\mathrm{d}}{\mathrm{d}r}G(\boldsymbol{r}) = \hat{\boldsymbol{r}}\, H_1(kr),$$

with $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ and $r = |\boldsymbol{r}|$. Each component of $\boldsymbol{g}$ is an outgoing solution of the Helmholtz equation.

The first term in (14) is a source at $\boldsymbol{r}_j$; the strength of the source (given by A_j) is unknown. The second term is a dipole at $\boldsymbol{r}_j$; the direction and strength of the dipole (given by $\boldsymbol{q}_j$) are unknown. The basic Foldy method assumes that $\boldsymbol{q}_j \equiv \mathbf{0}$. We remark that the approximation (14) was used successfully in [12, Appendix A] for scattering by an infinite grating of sound-hard circular cylinders.

For more detail, define polar coordinates R_j and Θ_j at $\boldsymbol{r}_j$, $\boldsymbol{r} = \boldsymbol{r}_j + R_j(\hat{\mathbf{\imath}}\cos\Theta_j + \hat{\mathbf{\jmath}}\sin\Theta_j)$, where $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ are unit vectors in the x and y directions, respectively. Then (14) becomes

$$A_j H_0(kR_j) + \{(\boldsymbol{q}_j \cdot \hat{\mathbf{\imath}})\cos\Theta_j + (\boldsymbol{q}_j \cdot \hat{\mathbf{\jmath}})\sin\Theta_j\} H_1(kR_j).$$

Next, we represent the total field as

$$u(\boldsymbol{r}) = u_{\mathrm{inc}}(\boldsymbol{r}) + \sum_{j=1}^{N}\{A_j G(\boldsymbol{r} - \boldsymbol{r}_j) + \boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j)\}. \qquad (15)$$

The field incident on $\mathcal{C}_n$ in the presence of all the other scatterers is

$$u_n(\boldsymbol{r}) \equiv u(\boldsymbol{r}) - A_n G(\boldsymbol{r} - \boldsymbol{r}_n) - \boldsymbol{q}_n \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_n) \qquad (16)$$
$$= u_{\mathrm{inc}}(\boldsymbol{r}) + \sum_{\substack{j=1\\ j\neq n}}^{N}\{A_j G(\boldsymbol{r} - \boldsymbol{r}_j) + \boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j)\}.$$

This "incident" field is scattered by $\mathcal{C}_n$. We characterize this process by

$$A_n = g u_n(\boldsymbol{r}_n) \quad \text{and} \quad \boldsymbol{q}_n = \mathbf{Q}\,\boldsymbol{v}_n(\boldsymbol{r}_n),$$

where

$$\boldsymbol{v}_n(\boldsymbol{r}) = k^{-1}\mathrm{grad}\, u_n. \qquad (17)$$

The quantity g is a scalar whereas $\mathbf{Q}$ is a 2×2 matrix. Thus, A_n is proportional to the value of the exciting field at $\boldsymbol{r}_n$, whereas $\boldsymbol{v}_n$ is related to the gradient of the exciting field at $\boldsymbol{r}_n$.

Low-frequency asymptotics for scattering by one sound-hard circle lead to good choices for g and $\mathbf{Q}$. Thus

$$g = -\frac{J_0'(ka)}{H_0'(ka)} \sim \frac{\pi}{4\mathrm{i}}(ka)^2, \quad \mathbf{Q} = -2g\mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix.

Evaluating (16) at $\boldsymbol{r}_n$ gives

$$\frac{1}{g}A_n = u_{\mathrm{inc}}(\boldsymbol{r}_n) + \sum_{\substack{j=1\\ j\neq n}}^{N}\{A_j G(\boldsymbol{R}_{nj}) + \boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{R}_{nj})\}, \qquad (18)$$

where $\boldsymbol{R}_{nj} = \boldsymbol{r}_n - \boldsymbol{r}_j$. Also, from (16) and (17),

$$\boldsymbol{v}_n(\boldsymbol{r}) = \boldsymbol{v}_{\mathrm{inc}}(\boldsymbol{r}) \qquad (19)$$
$$+ \sum_{\substack{j=1\\ j\neq n}}^{N}\{-A_j\,\boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j) + k^{-1}\mathrm{grad}\,[\boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j)]\},$$

where $\boldsymbol{v}_{\mathrm{inc}}(\boldsymbol{r}) = k^{-1}\mathrm{grad}\, u_{\mathrm{inc}}$. Direct calculation gives

$$(kR_{nj})^{-1}H_1(kR_{nj})\,\boldsymbol{q}_j - \widehat{\boldsymbol{R}}_{nj}(\boldsymbol{q}_j \cdot \widehat{\boldsymbol{R}}_{nj})H_2(kR_{nj})$$

for $k^{-1}\mathrm{grad}\,[\boldsymbol{q}_j \cdot \boldsymbol{g}(\boldsymbol{r} - \boldsymbol{r}_j)]$ at $\boldsymbol{r} = \boldsymbol{r}_n$, where $R_{nj} = |\boldsymbol{R}_{nj}|$ and $\widehat{\boldsymbol{R}}_{nj} = \boldsymbol{R}_{nj}/R_{nj}$. Hence, evaluating (19) at $\boldsymbol{r}_n$, we obtain

$$\mathbf{Q}^{-1}\boldsymbol{q}_n = \boldsymbol{v}_{\mathrm{inc}}(\boldsymbol{r}_n) + \sum_{\substack{j=1\\ j\neq n}}^{N}\left\{\frac{H_1(kR_{nj})}{kR_{nj}}\,\boldsymbol{q}_j - \widehat{\boldsymbol{R}}_{nj}(\boldsymbol{q}_j \cdot \widehat{\boldsymbol{R}}_{nj})H_2(kR_{nj}) - A_j\boldsymbol{g}(\boldsymbol{R}_{nj})\right\}. \qquad (20)$$

Equations (18) and (20) hold for $n = 1, 2, \ldots, N$. They give a system of linear algebraic equations for A_n and the two components of $\boldsymbol{q}_n$. For N scatterers, there are $3N$ equations for the $3N$ scalar unknowns.

5. Application to a ring of hard cylinders

We define the geometry as in §3. It is convenient to write $\boldsymbol{q}_j$ in terms of its radial and tangential components with respect to the ring. Let $\hat{\boldsymbol{\theta}}_j = \hat{\mathbf{j}}\cos\theta_j - \hat{\mathbf{i}}\sin\theta_j$ be a unit tangent vector, so that $\hat{\boldsymbol{r}}_j \cdot \hat{\boldsymbol{\theta}}_j = 0$. Write

$$\boldsymbol{q}_j = B_j \hat{\boldsymbol{r}}_j + C_j \hat{\boldsymbol{\theta}}_j,$$

so that the $3N$ unknowns are A_j, B_j and C_j, $j = 1, 2, \ldots, N$. We have

$$\begin{aligned} b^{-1}\boldsymbol{q}_j \cdot \boldsymbol{R}_{nj} &= (B_j \hat{\boldsymbol{r}}_j + C_j \hat{\boldsymbol{\theta}}_j) \cdot (\hat{\boldsymbol{r}}_n - \hat{\boldsymbol{r}}_j) \\ &= -(2b^2)^{-1} R_{nj}^2 B_j + C_j \sin\theta_{nj}, \end{aligned}$$

where we have used (5) and we have defined

$$\theta_{nj} = \theta_n - \theta_j = (n-j)h.$$

Hence

$$\boldsymbol{q}_j \cdot \widehat{\boldsymbol{R}}_{nj} = -(2b)^{-1} R_{nj} B_j + b R_{nj}^{-1} C_j \sin\theta_{nj}.$$

This will be used in (18) and (20). We will also form the inner product of (20) with $\hat{\boldsymbol{r}}_n$ and with $-\hat{\boldsymbol{\theta}}_n$. Thus, we require

$$\begin{aligned} \hat{\boldsymbol{r}}_n \cdot \boldsymbol{q}_j &= B_j \cos\theta_{nj} + C_j \sin\theta_{nj}, \\ \hat{\boldsymbol{\theta}}_n \cdot \boldsymbol{q}_j &= -B_j \sin\theta_{nj} + C_j \cos\theta_{nj}, \\ \hat{\boldsymbol{r}}_n \cdot \widehat{\boldsymbol{R}}_{nj} &= (2b)^{-1} R_{nj}, \\ \hat{\boldsymbol{\theta}}_n \cdot \widehat{\boldsymbol{R}}_{nj} &= b R_{nj}^{-1} \sin\theta_{nj}. \end{aligned}$$

Assembling all the pieces, we obtain the system

$$\sum_{j=1}^{N} \mathbf{K}_{n-j} \mathbf{x}_j = \mathbf{f}_n, \quad n = 1, 2, \ldots, N, \tag{21}$$

where $\mathbf{x}_j = (A_j, B_j, C_j)^T$,

$$\mathbf{f}_j = (-u_{\text{inc}}(\boldsymbol{r}_n),\, -\hat{\boldsymbol{r}}_n \cdot \boldsymbol{v}_{\text{inc}}(\boldsymbol{r}_n),\, \hat{\boldsymbol{\theta}}_n \cdot \boldsymbol{v}_{\text{inc}}(\boldsymbol{r}_n))^T$$

and $\mathbf{K}_j$ is a symmetric 3×3 matrix. In detail,

$$\mathbf{K}_0 = \mathbf{K}_N = \begin{pmatrix} -g^{-1} & 0 & 0 \\ 0 & (2g)^{-1} & 0 \\ 0 & 0 & -(2g)^{-1} \end{pmatrix}$$

and, for $j \neq 0 \bmod N$,

$$\mathbf{K}_j = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix},$$

with entries as follows:

$$\begin{aligned} K_{11} &= H_0, \quad K_{12} = -(2b)^{-1} R_j H_1, \\ K_{13} &= b R_j^{-1} H_1 \sin\theta_j, \\ K_{22} &= \frac{H_1}{kR_j}\cos\theta_j + H_2 \frac{R_j^2}{4b^2}, \\ K_{23} &= \frac{H_1}{kR_j}\sin\theta_j - \frac{1}{2} H_2 \sin\theta_j, \\ K_{33} &= -\frac{H_1}{kR_j}\cos\theta_j + H_2 \frac{b^2}{R_j^2}\sin^2\theta_j. \end{aligned}$$

All the Hankel functions have argument kR_j with $R_j = 2b|\sin(j\pi/N)|$. Evidently, $\mathbf{K}_j$ is N-periodic: $\mathbf{K}_{j+mN} = \mathbf{K}_j$, $m = \pm 1, \pm 2, \ldots$.

The system (21) gives $3N$ equations for $3N$ unknowns. Application of the discrete Fourier transform breaks the system into N 3×3 systems, one for each $\mathbf{x}_j$.

6. Discussion

This is a work in progress. The current intention is to develop the approach outlined in §5 so as to analyse the effects of letting N grow. One question is: how well does a ring of small vertical cylinders shield the interior of the ring? We hope to present results in this direction at the Workshop.

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