When no axisymmetric modes are trapped by a freely floating moonpool

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We study the coupled time-harmonic motion of an axisymmetric moonpool (toroidal, surface-piercing body) floating freely in infinitely deep water bounded above by a free surface. The surface tension is neglected and the water motion is assumed to be irrotational. The motion of both water and body is of small amplitude near equilibrium which allows us to apply the linearized model proposed by John [2]. We use it in the form of a coupled spectral problem; its two- and three-dimensional versions were developed in [4] and [7], respectively.

In the framework of this model, two theorems proved in [7] guarantee the absence of trapped modes in the presence of a single body freely floating in water of constant finite depth; it is either totally submerged or surface piercing, but the free surface is assumed to be connected in the latter case. On the other hand, single bodies and structures consisting of multiple bodies [either motionless or heaving] were constructed in [8] and [9] so that they trap axisymmetric wave modes in water of infinite depth. A characteristic feature of these bodies and structures, all of which have axisymmetric immersed parts and float freely, is that they divide the free surface into at least two connected components.

In the present note, our aim is to find frequency intervals within which no axisymmetric modes exist that are trapped by a freely floating moonpool provided its geometry satisfies the assumptions used in [5] to guarantee the absence of modes trapped by the same moonpool being fixed. The frequency intervals obtained here only partly coincide with those in [5] because an extra condition is due to the equation of body's motion.

1 Statement of the problem

We take the Cartesian coordinates $(\boldsymbol{x}, y)$ $[\boldsymbol{x} = (x_1, x_2)]$ so that the y-axis is directed upwards and the $\boldsymbol{x}$-plane coincides with the mean free surface. It divides $\widehat{B}$ — an axisymmetric toroidal domain occupied by a surface-piercing body in its equilibrium position — into two non-empty parts and $B = \widehat{B} \cap \mathbb{R}^3_-$ denotes the submerged part $[\mathbb{R}^3_- = \{(\boldsymbol{x}, y) : \boldsymbol{x} \in \mathbb{R}^2, y < 0\}]$. Furthermore, $W = \mathbb{R}^3_- \setminus \overline{B}$ is the water domain, $S = \partial \widehat{B} \cap \mathbb{R}^3_-$ is the wetted surface of the moonpool and $\boldsymbol{n}$ denotes the unit normal pointing to the exterior of W. Finally, the free surface $F = \partial W \setminus \overline{S}$ is the union $F_0 \cup F_\infty$; its circular part $F_0 = \{0 \leqslant |\boldsymbol{x}| < b, y = 0\}$ is separated from infinity by the annulus $D = \widehat{B} \cap \partial \mathbb{R}^3_-$ of finite width and F_∞ is the infinite part of the free surface outside of D.

In the general linearised setting (see [2]), the time-dependent motion is described by the following first-order variables: the real-valued velocity potential $\Phi(\boldsymbol{x}, y; t)$ and the vector $\boldsymbol{q}(t) \in \mathbb{R}^6$, characterising the motion of the body's centre of mass about its rest position $\big(\boldsymbol{x}^{(0)}, y^{(0)}\big)$. The horizontal and vertical displacements are q_1, q_2 and q_4, respectively, whereas q_3 and q_5, q_6 are the angles of rotation about the axes that go through the centre of mass and are parallel to the y and x_1, x_2 axes, respectively.

We do not formulate the problem for Φ and $\boldsymbol{q}$ (see [7] for its condensed form), but assume that the motion is time-harmonic with the radian frequency $\omega > 0$, in which case $\big(\Phi(\boldsymbol{x}, y, t), \boldsymbol{q}(t)\big) = \mathrm{Re}\big\{\mathrm{e}^{-\mathrm{i}\omega t}\big(\varphi(\boldsymbol{x}, y), \boldsymbol{\chi}\big)\big\}$. Then the bounded complex-valued function φ and $\boldsymbol{\chi} \in \mathbb{C}^6$ must satisfy the following problem:

$$\nabla^2 \varphi = 0 \ \text{ in } W; \ \ \partial_y \varphi - \nu \varphi = 0 \ \text{ on } F; \ \ \partial_{\boldsymbol{n}} \varphi = -\mathrm{i}\omega\, \boldsymbol{n}^\mathsf{T} \boldsymbol{D}_0 \boldsymbol{\chi} \ \text{ on } S; \tag{1}$$

$$\omega^2 \boldsymbol{E} \boldsymbol{\chi} = -\mathrm{i}\omega \int_S \varphi \boldsymbol{D}_0^\mathsf{T} \boldsymbol{n}\, \mathrm{d}s + g \boldsymbol{K} \boldsymbol{\chi}; \quad \int_{W \cap \{|\boldsymbol{x}| = a\}} \big|\partial_{|\boldsymbol{x}|} \varphi - \mathrm{i}\nu\varphi\big|^2 \mathrm{d}s = o(1) \ \text{ as } \ a \to \infty. \tag{2}$$

Here ω is the spectral parameter [$\nu = \omega^2/g$ for the sake of brevity], which is sought together with the eigenvector $(\varphi, \boldsymbol{\chi})$; also the following notation is used: $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_y)$ is the spatial gradient, g is the acceleration due to gravity acting in the direction opposite to the y-axis, $^\mathsf{T}$ is the symbol of matrix transposition; $\boldsymbol{D}_0 = \boldsymbol{D}\big(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\big)$ is an auxiliary 3×6 matrix, where $\boldsymbol{D}(\boldsymbol{x}, y) = \begin{bmatrix} 1 & 0 & x_2 & 0 & 0 & -y \\ 0 & 1 & -x_1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & -x_2 & x_1 \end{bmatrix}$. In the equation of body's motion [the first condition (2)], the 6×6 matrices $\boldsymbol{E}$ and $\boldsymbol{K}$ are defined as follows:

$\boldsymbol{E} = \rho_0^{-1} \int_{\widehat{B}} \rho\,(\boldsymbol{x}, y)\, \boldsymbol{D}_0^{\mathsf{T}}(\boldsymbol{x}, y)\, \boldsymbol{D}_0(\boldsymbol{x}, y)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y$, where $\rho\,(\boldsymbol{x}, y) \geqslant 0$ is the distribution of density within the body and $\rho_0 > 0$ is the constant density of water;

$$\boldsymbol{K} = \begin{pmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{O}_3 & \boldsymbol{K}' \end{pmatrix}, \quad \text{where} \quad \boldsymbol{K}' = \begin{pmatrix} I^D & I_2^D & -I_1^D \\ I_2^D & I_{22}^D + I_y^B & -I_{12}^D \\ -I_1^D & -I_{12}^D & I_{11}^D + I_y^B \end{pmatrix}, \quad I^D = \int_D \mathrm{d}\boldsymbol{x}, \quad I_y^B = \int_B \big(y - y^{(0)}\big)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y,$$

$$I_i^D = \int_D \big(x_i - x_i^{(0)}\big)\, \mathrm{d}\boldsymbol{x}, \quad I_{ij}^D = \int_D \big(x_i - x_i^{(0)}\big)\big(x_j - x_j^{(0)}\big)\, \mathrm{d}\boldsymbol{x}, \;\; i, j = 1, 2, \quad \text{and } \mathbb{O}_3 \text{ is the } 3\times 3 \text{ null matrix.}$$

The elements of the mass-inertia matrix $\boldsymbol{E}$ are various moments of the whole body $\widehat{B}$ and this matrix is symmetric and positive definite; the symmetric matrix $\boldsymbol{K}$ is related to buoyancy (see [2, 7]).

Along with relations (1) and (2), the following subsidiary conditions must hold [they concern the equilibrium position of the floating body and its stability]: • $\rho_0^{-1} \int_{\widehat{B}} \rho\,(\boldsymbol{x}, y)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y = \int_B \mathrm{d}\boldsymbol{x}\, \mathrm{d}y$ [Archimedes' law];

• $\int_B \big(x_i - x_i^{(0)}\big)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y = 0$, $i = 1, 2$ [the center of buoyancy lies on the same vertical line as the centre of mass];

• $\boldsymbol{K}'$ is a positive definite matrix which implies that the body's equilibrium position is stable, [2, § 2.4].

2 Definition of a trapped mode

First we list some properties of φ under the assumption that $\widehat{B}$ is an axisymmetric freely floating moonpool with $x_1^{(0)} = x_2^{(0)} = 0$. The boundedness of φ implies that $\nabla\varphi$ decays as $y \to -\infty$, whereas the radiation condition [the second condition (2)] guarantees that φ describes outgoing waves at infinity (see [3]). In the same way as in [7], one proves the following assertion about the energy of (φ, χ) satisfying problem (1) and (2):

The first component φ belongs to the space $H^1(W)$ and $\int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} < \infty$, that is, the kinetic and potential energy of the water motion is finite. The following equality expresses the equipartition of energy of the coupled motion

$$\int_W |\nabla\varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y + \omega^2 \overline{\chi}^{\mathsf{T}} \boldsymbol{E} \chi = \nu \int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} + g\, \overline{\chi}^{\mathsf{T}} \boldsymbol{K} \chi. \tag{3}$$

Definition. Let the conditions concerning the equilibrium position of the moonpool $\widehat{B}$ hold, then a non-trivial pair (φ, χ) belonging to $H^1(W) \times \mathbb{C}^6$ is called a *trapped mode* provided relations (1) and the first relation (2) are satisfied for some value of ω; the latter is called a *trapping frequency*.

Since $\varphi \in H^1(W)$ relations (1) must be understood as the integral identity

$$\int_W \nabla\varphi \nabla\psi\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y = \nu \int_F \varphi\, \psi\, \mathrm{d}\boldsymbol{x} - \mathrm{i}\omega \int_S \psi\, \boldsymbol{n}^{\mathsf{T}} \boldsymbol{D}_0\, \chi\, \mathrm{d}s, \tag{4}$$

which involves only the first-order derivatives of φ; here ψ is an arbitrary smooth function having a compact support in $\overline{W}$.

Below we restrict our considerations to axisymmetrc trapped modes, that is, $\chi = (0, 0, 0, H, 0, 0)$, whereas $\varphi = \varphi(|\boldsymbol{x}|, y)$. We recall that several examples of geometries of bodies [and also of multi-body structures] that trap modes with $H = 0$ [$\widehat{B}$ is motionless] and $H \neq 0$ [$\widehat{B}$ is heaving] were constructed in [8] and [9], respectively.

3 Conditions guaranteeing the absence of axisymmetric trapped modes

We recall that a body satisfies John's condition [3] if no point of its wetted surface lies on vertical lines going through the free surface (see Fig. 1 in [5], illustrating the case of an axisymmetric moonpool). This condition is essential for proving the following assertion about the absence of axisymmetric trapped modes.

Let an axisymmetric freely floating moonpool satisfy John's condition. If $(\varphi, \chi) \in H^1(W) \times \mathbb{C}^6$ is an axisymmetric solution of problem that consists of the first relation (2) *and identity* (4)*, then this solution vanishes identically provided the following two conditions are fulfilled:*

(i) *the inequalities $j_{0,m} \leqslant \nu b \leqslant j_{1,m}$ hold for some $m = 1, 2, \ldots$, where $j_{0,m}$ $(j_{1,m})$ is the m-th zero of the Bessel function J_0 (J_1, respectively*);

(ii) $\omega^2 > \lambda_*$*, where $\lambda_* > 0$ is the largest λ such that* $\det(\lambda \boldsymbol{E} - g\boldsymbol{K}) = 0$.

What follows outlines main points of the proof. As in the case of a fixed moonpool (see [5]), we assume that there exists a non-trivial φ and define the following function which was originally introduced by John [3] under the assumption that $F_0 = \emptyset$:

$$w(|\boldsymbol{x}|) = \int_{-\infty}^{0} \varphi(|\boldsymbol{x}|, y)\, \mathrm{e}^{\nu y}\, \mathrm{d}y. \tag{5}$$

Then the Laplace equation and the free surface boundary condition [the first and second relations (1)] imply that

$$w_{x_1 x_1} + w_{x_2 x_2} + \nu^2 w = 0 \quad \text{in } F = F_0 \cup F_\infty. \tag{6}$$

Since φ decays at infinity, this gives that $w \equiv 0$ in F_∞ (see [3], p. 78), a consequence of which is the inequality

$$\nu \int_{F_\infty} |\varphi|^2\, \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{2} \int_{W_\infty} |\partial_y \varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y \leqslant \frac{1}{2} \int_{W_\infty} |\nabla \varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y. \tag{7}$$

Furthermore, (6) yields that $w(|\boldsymbol{x}|) = C J_0(\nu |\boldsymbol{x}|)$ in F_0 because φ is non-singular on $|\boldsymbol{x}| = 0$; here C is, generally speaking, a non-zero constant. Thus, we get that the following equalities are valid for $|\boldsymbol{x}| < b$:

$$C J_0(\nu |\boldsymbol{x}|) = \int_{-\infty}^{0} \varphi(|\boldsymbol{x}|, y)\, \mathrm{e}^{\nu y}\, \mathrm{d}y, \qquad \nu C J_1(\nu |\boldsymbol{x}|) = -\int_{-\infty}^{0} \varphi_{|\boldsymbol{x}|}(|\boldsymbol{x}|, y)\, \mathrm{e}^{\nu y}\, \mathrm{d}y,$$

$$\varphi(|\boldsymbol{x}|, 0) = \nu C J_0(\omega^2 |\boldsymbol{x}|) - \int_{-\infty}^{0} \varphi_y(|\boldsymbol{x}|, y)\, \mathrm{e}^{\nu y}\, \mathrm{d}y.$$

Here the second and third equalities follow from the first one by virtue of differentiation and integration by parts, respectively. After some manipulations with inequalities obtained by squaring these relations and applying the Schwarz inequality to the integrals (see [5], pp. 570–572), one arrives at the inequality

$$\nu \int_{F_0} |\varphi|^2\, \mathrm{d}\boldsymbol{x} \leqslant \int_{W_0} |\nabla \varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y \tag{8}$$

under the assumption that νb satisfies condition (i). From this inequality and (7) we get that

$$\nu \int_{F} |\varphi|^2\, \mathrm{d}\boldsymbol{x} < \int_{W} |\nabla \varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y. \tag{9}$$

Then the equipartition of energy [equality (3)] gives

$$\omega^2 \overline{\chi}^{\mathsf{T}} \boldsymbol{E} \chi - g \overline{\chi}^{\mathsf{T}} \boldsymbol{K} \chi < 0, \tag{10}$$

which is incompatible with (ii) unless χ vanishes, but this is impossible because then (9) contradicts (3) for a non-trivial φ. Hence the problem under consideration has only a trivial axisymmetric solution.

4 Discussion

• If the free surface F is connected and $N \geqslant 1$ bodies float freely each satisfying John's condition, then condition (i) is unnecessary for obtaining inequality (9). However, condition (ii) must hold for each body and λ_* in this case is equal to the largest λ_j, where the latter is the largest λ such that $\det(\lambda \boldsymbol{E}_j - g \boldsymbol{K}_j) = 0$, $j = 1, \ldots, N$.

• If a moonpool is fixed, then the uniqueness follows directly from inequality (9) because it contradicts (3) since the latter does not contain the two terms depending on χ.

• In the case when W is bounded below by a horizontal bottom $\{y = -d\}$ with $d > \max\{-y : (x, y) \in S\}$, the uniqueness theorem similar to that obtained in § 3 for deep water is also true, but the non-dimensional parameter νb in condition (i) must be changed to $k_0 b$, where k_0 is the unique positive root of $k_0 \tanh k_0 d = \nu$. In the proof, one has to use $w(|\boldsymbol{x}|) = \int_{-d}^{0} \varphi(|\boldsymbol{x}|, y) \cosh k_0 y\, \mathrm{d}y$ instead of (5) when deriving inequalities (7) and (8) for the axisymmetric mode φ.

• The above theorem can be improved by allowing the submerged part B of a moonpool to be bulbous on the side directed to infinity. Namely, let B lies outside the cylinder $\{|\boldsymbol{x}| = b\}$ and within the cone whose generator going through the waterline $\overline{S} \cap \overline{F}_\infty$ forms an angle less than $38°$ with the negative y-axis. Then the inequality

$$\nu \int_{F_\infty} |\varphi|^2\, \mathrm{d}\boldsymbol{x} \leqslant \int_{W_\infty} |\nabla \varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y$$

holds as is shown in [10]; here W_∞ is the part of $\mathbb{R}^3_-$ lying outside the cone described above. Combing the last inequality [instead of (7)] and (8) [a consequence of condition (i)], one obtains the theorem for a bulbous axisymmetric moonpool. Below we explain why condition (i) which plays a key role in our considerations is incompatible with a moonpool's geometry bulbous on the side directed to W_0.

• A result similar to the above theorem is true for an azimuthal mode of the form $\varphi_n(|\boldsymbol{x}|, y) \cos n\theta$, where θ is the polar angle in the $\boldsymbol{x}$-plane and $n = 1, 2, \ldots$, in which case condition (i) must be changed to the following one. *The inequalities $j_{n,m} \leqslant \nu b \leqslant j'_{n,m}$ hold for some $m = 1, 2, \ldots$; here $j_{n,m}$ ($j'_{n,m}$) is the m-th zero of the Bessel function J_n (J'_n, respectively).* According to formulae 9.5.12 and 9.5.13 in [1] [they give the asymptotics of these zeroes as $m \to \infty$], the intervals in νb, for which there are no trapped modes with any azimuthal number, are asymptotically of length $\pi/2$.

• For values of b, belonging to a certain interval adjacent to $j_{1,m}/\nu$ and located on the left of it, there exist freely floating moonpools both heaving and motionless [the latter can be also treated as fixed] trapping axisymmetric modes and enclosing the circular free surface of radius b. When $m = 1$ such values of b lie between $|\mathring{\boldsymbol{x}}| \in (0, j_{1,1}/\nu)$ and $j_{1,1}/\nu$ as follows from Proposition 5 (c) in [8]; here $|\mathring{\boldsymbol{x}}|$ stands for the unique positive zero of the trace $\psi(\nu|\boldsymbol{x}|, 0)$ of the stream function that has its singularity at $(j_{1,1}, 0)$. If the singularity is at $(j_{1,m}, 0)$, then the behaviour of the left end of the interval occupied by values νb corresponding to trapping moonpools is as follows: it tends to $j_{0,m}$ as m goes to infinity. Hence, the intervals in which trapped modes exist are also asymptotically of length $\pi/2$.

• Results similar to those in § 3 are valid for the properly formulated two-dimensional problem. Let a pair of infinitely long surface-piercing cylinders have a vertical cross-section symmetric about the y-axis (see Fig. 4.7 in [6]). In this case, it is natural to consider symmetric and antisymmetric modes of waves as is shown in [6], § 4.2.2, for fixed cylinders. Indeed, conditions guaranteeing the absence of trapped modes of each type are given there. It occurs that the same results are valid when the cylinders have symmetric density distributions and float freely. In particular, for the parameter νb [here $2b$ is the spacing between the cylinders] there are segments, where antisymmetric and symmetric modes are absent and these segments are interlacing. However, the additional restriction which is unnecessary for fixed cylinders must be imposed when cylinders are freely floating; it says that frequencies must be sufficiently large [cf. condition (ii)]. It is also worth mentioning that if $\nu b = \pi m/2$ and $m > 0$ is sufficiently large, then there are no trapped modes at all in deep water.

5 Conjecture

Given the proof of a theorem guaranteeing the uniqueness of a solution to the linearised problem about time-harmonic water waves in the presence of a fixed obstacle, then this proof admits amendments transforming it into the proof of an analogous theorem for the same obstacle floating freely with additional restrictions on the non-trapping frequencies [they must be sufficiently large; see condition (ii) above] and, in some cases, on body's geometry and on the type of non-trapping modes.

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