The interaction of a waves with a Submerged Very Large Elastic Plate

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1 Introduction

Last year at the IWWWFB in Osaka Porter et al [1] discussed the phenomenon of cloaking. Among others results for a stiff plate underneath the free surface was considered. I asked myself the question whether for a flexible plate underneath the free surface the same phenomenon could be observed. In this presentation we extend the 'mode' method as derived for flexible plates at the free surface. It is well known, see for instance Hermans [2], that at certain frequencies the transmission coefficient may have an absolute value equal to unity $|T| = 1$. We will check by means of numerical calculations whether at such a point the phase shift equals zero, hence $T = 1$. In [2] a small phase shift is present at the frequency where $|T| = 1$.
In this abstract we extend the existing 'mode' method to the submerged flexible plate situation. We first consider a flexible plate and then a fixed rigid plate.

2 Flexible plate

We consider the two-dimensional interaction of an incident wave with a Very Large Flexible Platform (VLFP) with zero thickness located at finite depth $y = y_0$. We want to make use of the Green's theorem as we have done in the case where the platform is at the free surface.

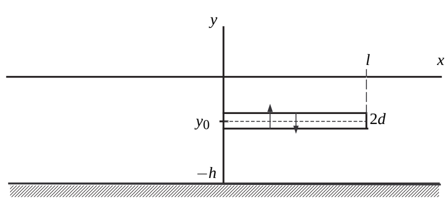


Figure 1: configuration

To do so we start with a plate of finite thickness, $2d$, as depicted in figure (1) and later take the limit $d \to 0$. The advantage is that we can make use of two different forms of the Green's function for points above or underneath the platform. The fluid is ideal, so we introduce the velocity potential $\mathbf{V}(\mathbf{x},t) = \nabla\Phi(\mathbf{x},t)$, where $\mathbf{V}(\mathbf{x},t)$ is the fluid velocity vector. Hence $\Phi(\mathbf{x},t)$is a solution of the Laplace equation $\Delta\Phi = 0$ in the fluid, together with the linearized kinematic condition, $\Phi_y = \tilde{v}_t$, and dynamic condition, $p/\rho = -\Phi_t - g\tilde{v}$, at the mean water surface $y = 0$, where $\tilde{v}(x,t)$ denotes the free surface elevation, and ρ is the density of the water.

The linearized free surface condition outside the platform, $y = 0$ and $x \in \mathcal{F}$, becomes:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial y} = 0. \tag{1}$$

We assume that the velocity potential is a time-harmonic wave function, $\Phi(\mathbf{x},t) = \phi(\mathbf{x})\ \mathrm{e}^{-\mathrm{i}\omega t}$. The potential of the undisturbed incident wave is given by:

$$\phi^{\mathrm{inc}}(\mathbf{x}) = \frac{g\zeta_\infty}{\mathrm{i}\omega}\frac{\cosh(k_0(y+h))}{\cosh(k_0 h)}\exp(\mathrm{i}k_0 x) \tag{2}$$

where ζ_∞ is the wave height in the original coordinate system, ω the frequency, while the wave number k_0 is the positive real solution of the dispersion relation, $k_0\tanh(k_0 h) = K$, for finite water depth.
A crucial step is the choice of the Green's function. It is possible to derive the Green's function $\mathcal{G}(x,y;\xi,\eta)$ by means of a Fourier transform with respect to the x-coordinate. It has the form:

$$\mathcal{G}^-(x,y;\xi,\eta) = \int_{\mathcal{L}'}\frac{1}{\gamma}\frac{K\sinh\gamma y + \gamma\cosh\gamma y}{K\cosh\gamma h - \gamma\sinh\gamma h}\cosh\gamma(\eta+h)\ \mathrm{e}^{\mathrm{i}\gamma(x-\xi)}\,\mathrm{d}\gamma \quad \text{for } y > \eta \tag{3}$$

and

$$\mathcal{G}^+(x,y;\xi,\eta) = \int_{\mathcal{L}'}\frac{1}{\gamma}\frac{K\sinh\gamma\eta + \gamma\cosh\gamma\eta}{K\cosh\gamma h - \gamma\sinh\gamma h}\cosh\gamma(y+h)\ \mathrm{e}^{\mathrm{i}\gamma(x-\xi)}\,\mathrm{d}\gamma \quad \text{for } y < \eta \tag{4}$$

The Green's function obeys the free surface boundary condition. The expression for the total potential $\phi^{\pm}$ becomes for $y > y_0 + d$ and for $y < y_0 - d$ resp.

$$2\pi\phi^{\pm}(x,y) = 2\pi\phi^{\text{inc}}(x,y) - \int_0^l \left(\phi^{+}(\xi,\eta)\frac{\partial\mathcal{G}^{\mp}(x,y;\xi,\eta)}{\partial\eta} - \frac{\partial\phi^{+}(\xi,\eta)}{\partial\eta}\mathcal{G}^{\mp}(x,y;\xi,\eta)\right)\Bigg|_{\eta=y_0+d} - \left(\phi^{-}(\xi,\eta)\frac{\partial\mathcal{G}^{\mp}(x,y;\xi,\eta)}{\partial\eta} - \frac{\partial\phi^{-}(\xi,\eta)}{\partial\eta}\mathcal{G}^{\mp}(x,y;\xi,\eta)\right)\Bigg|_{\eta=y_0-d} \mathrm{d}\xi. \tag{5}$$

where we ignored the contributions of the endpoints.
To describe the vertical deflection $\tilde{v}(x,t)$ of the platform, we apply the isotropic thin-plate theory and use the kinematic and dynamic condition to arrive at the following equation for ϕ at $y = y_0$ in the platform area $x \in \mathcal{P}$:

$$\left\{\mathcal{D}\frac{\partial^4}{\partial x^4} - \mu\right\}\frac{\partial\phi}{\partial y} = K(\phi^{-} - \phi^{+}), \tag{6}$$

where we used following parameters: $K = \frac{\omega^2}{g}$, $\mu = \frac{m\omega^2}{\rho g}$, $\mathcal{D} = \frac{D}{\rho g}$. Here m denotes the mass per length and D the flexural rigidity.
To apply the mode expansion we have to expand the potentials $\phi^{\pm}$ and the vertical velocity ϕ_y,and finally let d tend to zero, so we may pursue with this formulation.
However if we take $d = 0$ and differentiate (5) with respect to y we obtain the same result more directly. At first a hyper-singular equation does not look tractable. But it turns out that the equation of motion of the plate makes it possible to expand the vertical velocity at $y = y_0$ only.

$$2\pi\frac{\partial\phi}{\partial y} = 2\pi\frac{\partial\phi^{\text{inc}}(x,y)}{\partial y} + \frac{1}{K}\int_0^l \left|\left\{\left(\mathcal{D}\frac{\partial^4}{\partial\xi^4} - \mu\right)\frac{\partial\phi}{\partial\eta}\right\}\frac{\partial^2\mathcal{G}(x,y;\xi,\eta)}{\partial\eta\partial y}\right|_{\eta=y_0} \mathrm{d}\xi. \tag{7}$$

where

$$\frac{\partial^2\mathcal{G}(x,y;\xi,\eta)}{\partial\eta\partial y} = \int_{\mathcal{L}'}\gamma\,\frac{K\cosh\gamma y + \gamma\sinh\gamma y}{K\cosh\gamma h - \gamma\sinh\gamma h}\sinh\gamma(\eta+h)\;\mathrm{e}^{\mathrm{i}\gamma(x-\xi)}\,\mathrm{d}\gamma \tag{8}$$

We now introduce the expansions for the vertical velocity at the surface of the plate

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \sum_{n=0}^{N+1} a_n\,\mathrm{e}^{\mathrm{i}\kappa_n x} + b_n\,\mathrm{e}^{\mathrm{i}\kappa_n(l-x)} \tag{9}$$

We now can integrate with respect to ξ in (7) and let y tend to y_0. We consider $l = +\infty$ first and add some artificial damping to make the integrals with respect to ξ converge and work out the integral for $y > y_0$ and take $y = y_0$ afterwards.

$$\int_0^\infty\left\{\left(\mathcal{D}\frac{\partial^4}{\partial\xi^4} - \mu\right)\frac{\partial\phi}{\partial\eta}\right\}\frac{\partial^2\mathcal{G}(x,y;\xi,\eta)}{\partial\eta\partial y}\,\mathrm{d}\xi = \frac{a}{\mathrm{i}}(\mathcal{D}\kappa^4 - \mu)\int_{\mathcal{L}'}\gamma\frac{(K\cosh\gamma y + \gamma\sinh\gamma y)\sinh\gamma(y_0+h)}{(K\cosh\gamma h - \gamma\sinh\gamma h)(\gamma-\kappa)}\,\mathrm{e}^{\mathrm{i}\gamma x}\,\mathrm{d}\gamma =$$
$$2\pi a(\mathcal{D}\kappa^4 - \mu)\left(\kappa\frac{K\cosh\kappa y + \kappa\sinh\kappa y}{K\cosh\kappa h - \kappa\sinh\kappa h}\sinh\kappa(y_0+h)\,\mathrm{e}^{\mathrm{i}\kappa x} + \sum_{j=0}^{N-1}\frac{k_j(K\cosh k_j y + k_j\sinh k_j y)}{\frac{\mathrm{d}}{\mathrm{d}\gamma}(K\cosh\gamma h - \gamma\sinh\gamma h)|_{\gamma=k_j}}\frac{\sinh k_j(y_0+h)}{k_j-\kappa}\,\mathrm{e}^{\mathrm{i}k_j x}\right). \tag{10}$$

If we take the coefficient of $\exp(\mathrm{i}\kappa x)$ in (7) equal to zero we arrive at the following dispersion relation for κ

$$\kappa(\mathcal{D}\kappa^4 - \mu)\sinh\kappa(y_0+h) = -K\frac{\kappa\sinh\kappa h - K\cosh\kappa h}{\kappa\sinh\kappa y_0 + K\cosh\kappa y_0} \tag{11}$$

If we take $y_0 = 0$ we recover the dispersion relation for the plate at the free surface.

$$\kappa(\mathcal{D}\kappa^4 - \mu + 1)\tanh\kappa h = K \tag{12}$$

The terms with $\exp(\mathrm{i}k_j x)$ will give us a set of equations for the unknown coefficients $A_n = \frac{-a_n}{\mathrm{i}\omega}$ and $B_n = \frac{-b_n}{\mathrm{i}\omega}$ determining the deflection of the finite platform.

$$w(x) = \sum_{n=0}^{N+1} A_n\,\mathrm{e}^{\mathrm{i}\kappa_n x} + B_n\,\mathrm{e}^{\mathrm{i}\kappa_n(l-x)} \tag{13}$$

We have $2N$ equations of the $2N+4$ unknowns A_n and B_n:

$$\begin{aligned}
&\sum_{n=0}^{N+1}(\mathcal{D}\kappa_n^4-\mu)\left[\frac{A_n}{\kappa_n-k_i}-\frac{B_n\mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n+k_i}\right]=-\delta_{0i}\frac{\sinh k_0h}{\sinh k_0(y_0+h)}\frac{(K^2h-K-k_0^2h)}{Kk_0}\\
&\sum_{n=0}^{N+1}(\mathcal{D}\kappa_n^4-\mu)\left[\frac{A_n\mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n+k_i}-\frac{B_n}{\kappa_n-k_i}\right]=0.
\end{aligned}\tag{14}$$

for $i=0,\cdots,N-1$. The conditions at the endpoints of the platform $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=\frac{\mathrm{d}^3w}{\mathrm{d}x^3}=0$ result in four equations.

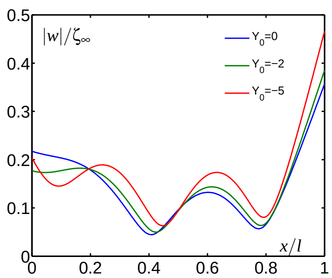


Figure 2: Deflection for different values of submergence

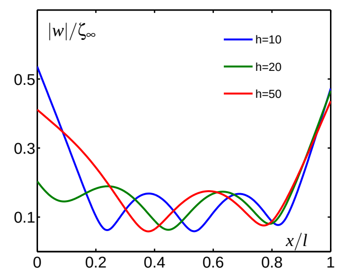


Figure 3: Deflection for different values of water depth

In figure (2) some results are shown for the deflection $|w|/\zeta_\infty$ of the plate with water depth $h=20$ m. , $l=300$ m. , $\mathcal{D}=10^7$ m. and $\lambda_0=2\pi/K=60$ m. for various values of submergence y_0. In figure (3) we have chosen a fixed value of submergence $y_0=-5$ m. and different values of water depth.

If we take $d=0$ in the expression for ϕ^+ in (5) we obtain the reflection and transmission coefficients.

$$R=\frac{Kk_0\sinh(k_0(y_0+h))}{(K(1-Kh)+k_0^2h)\sinh(k_0h}\sum_{n=0}^{N+1}(\mathcal{D}\kappa_n^4-\mu)\left[\frac{A_n}{k_0+\kappa_n}\left(\mathrm{e}^{i(k_0+\kappa_n)l}-1\right)+\frac{B_n}{k_0-\kappa_n}\left(\mathrm{e}^{ik_0l}-\mathrm{e}^{i\kappa_nl}\right)\right]\tag{15}$$

and

$$T=\frac{Kk_0\sinh(k_0(y_0+h))}{(K(1-Kh)+k_0^2h)\sinh(k_0h)}\sum_{n=0}^{N+1}(\mathcal{D}\kappa_n^4-\mu)\left[\frac{A_n}{k_0-\kappa_n}\left(\mathrm{e}^{-i(k_0-\kappa_n)l}-1\right)+\frac{B_n}{k_0+\kappa_n}\left(\mathrm{e}^{-ik_0l}-\mathrm{e}^{i\kappa_nl}\right)\right]+1\tag{16}$$

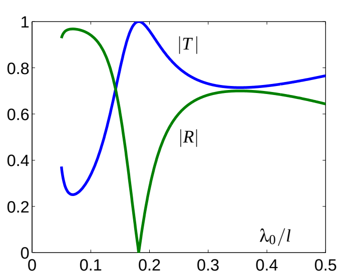


Figure 4: $\mathcal{D}=10^7$ m.

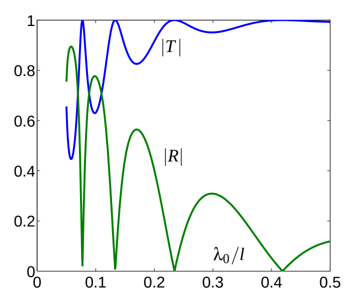


Figure 5: $\mathcal{D}=10^5$ m.

In figure (4) and (5) the reflection and Transmission coefficients are shown for two values of the flexural rigidity, $\mathcal{D}=10^7$ m. and $=10^5$ m. , at water depth $h=20$ m. and submergence of the plate $y_0=-5$ m. We study the points where $|T|=1$, it is more clear where $|R|=0$, in detail to see whether cloaking occurs in these points. We first notice that the relation $|R|^2+|T|^2=1$ very accurately.

The computations show that in none of these points $T=1$ exactly. In some points we find $\Re T$ close to -1 and in the other points where $\Re T$ is close to $+1$ the value of $\Im T\neq 0$.

3 Fixed rigid plate

If we consider a fixed rigid plate at $y=y_0$, thickness $d=0$. We notice that in the case of $\mathcal{D}\to\infty$ we find from the dispersion relation for the flexible plate (11) that either $\sinh\sigma(y_0+h)=0$ or $\kappa\sinh\kappa y_0+K\cosh\kappa y_0=0$, so we expect that we can use these realtions. To find proper relations for the modes we study the following hyper-singular equation obtained by means of we differentiation of (5) with respect to y, and take $y=y_0$.

$$2\pi\frac{\partial\phi^{\mathrm{inc}}(x,y)}{\partial y}=\int_0^l\left(\phi^+(\xi,\eta)-\phi^-(\xi,\eta)\right)\frac{\partial^2\mathcal{G}(x,y;\xi,\eta)}{\partial\eta\partial y}\,\mathrm{d}\xi.\tag{17}$$

This is a hyper-singular integral equation for the potential jump along the plate. We can solve this equation by means of the mode expansion method by introduction of:

$$\phi^+ - \phi^- = \sum_{n=0}^{N-1} \left(a_n\, e^{i\kappa_n x} + c_n\, e^{i\kappa_n (l-x)} \right) + \sum_{n=0}^{N-1} \left(b_n\, e^{i\sigma_n x} + d_n\, e^{i\sigma_n (l-x)} \right). \tag{18}$$

Using (10) it is easy to show that κ_n and σ_n are solutions of:

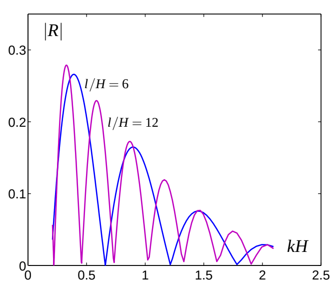
$$\kappa \sinh \kappa y_0 + K \cosh \kappa y_0 = 0, \qquad \sinh \sigma(y_0 + h) = 0. \tag{19}$$

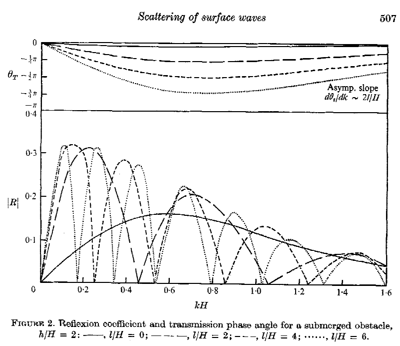
So the solution consists of a combination of the eigen-modes of the flow above and below the plate resp. as expected. In [3] we studied the case of a rigid plate at the free surface, in that case only the the second condition of (19) played a role.

For the $2N$ coefficients in (18) we obtain, for $i = 0, \cdots, 2N-1$ the following set of equations

$$\begin{aligned} &\sum_{n=0}^{N-1} \left[\frac{a_n}{\kappa_n - k_i} - \frac{c_n e^{i\kappa_n l}}{\kappa_n + k_i} \right] + \sum_{n=0}^{N-1} \left[\frac{b_n}{\sigma_n - k_i} - \frac{d_n e^{i\sigma_n l}}{\sigma_n + k_i} \right] = -\delta_{0i} \frac{\sinh k_0 h}{\sinh k_0 (y_0 + h)} \frac{(K^2 h + K - k_0^2 h)}{K k_0} \\ &\sum_{n=0}^{N-1} \left[\frac{a_n e^{i\kappa_n l}}{\kappa_n + k_i} - \frac{c_n}{\kappa_n - k_i} \right] + \sum_{n=0}^{N-1} \left[\frac{b_n e^{i\sigma_n l}}{\sigma_n + k_i} - \frac{d_n}{\sigma_n - k_i} \right] = 0. \end{aligned} \tag{20}$$

The transmission and reflection coefficients are obtained accordingly.





In a paper published in 1969 Mei and Black [5] studied the interaction of waves with fixed docks and object on the bottom. The figure shown is copied from that paper. In the figure next to it the plate is positioned at the same level as the top surface of the submerged object. The parameter H is the distance to the bottom.

4 Conclusions

We have extended the 'mode'method to the case of a flexible and fixed plate underneath the free surface. In contrast to the free surface plate one uses the hyper-singular integral equation for the vertical velocity instead of the equation for the potential function. The reflection coefficient computed by means of this method does not show frequencies where cloaking occurs. In [6] it is descibed how to extend the method to the configurations described in [5].

References

[1] D.V. Evans, M. McIver and R. Porter, Transparency of structures in water waves, *Proc. of the 29th International Workshop on Water Waves and Floating Bodies*, 2014, 37-41.

[2] A.J. Hermans, Free-surface wave interaction with a thick flexible dock or very large floating platform, *Journal of Engineering Mathematics* **58**, 2007, 77-90.

[3] A.J. Hermans, Interaction of free-surface waves with a floating dock, *Journal of Engineering Mathematics* **45**, 2003, 39-53.

[4] J.V. Wehausen and E.V. Laitone, Surface waves. *Encyclopedia of Physics* 9, Springer-Verlag, Berlin, (1960) 446-814, also http://www.coe.berkeley.edu/SurfaceWaves.

[5] C.C. Mei and J.L. Black, Scattering of surface waves by rectangular obstacles in waters of finite depth, *Journal of fluid mechanics*, **38**, 1969, 499-511.

[6] A.J. Hermans, Free-surface wave interaction with a thick flexible dock or very large floating platform, *Journal of Engineering Mathematics*, **58**, 2007, 7790.