Second-order hydroelastic behavior of a flexible circular plate in monochromatic waves

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Introduction

The hydroelastic behavior of a flexible circular plate with a concentric hole floating on the free surface is considered. The plate has zero thickness and homogeneous characteristics (constant density, constant flexural rigidity). It is also assumed that the fluid is the prefect fluid and the flow has irrotational characteristic. Furthermore, the thin plate theory is adopted to express the plate deflection. The right-handed coordinate system is introduced with $z = 0$ the undisturbed free surface. The bottom is assumed to be horizontal at $z = -h$. The incident wave propagates along positive x-axis. The basic configuration is shown in Figure 1.

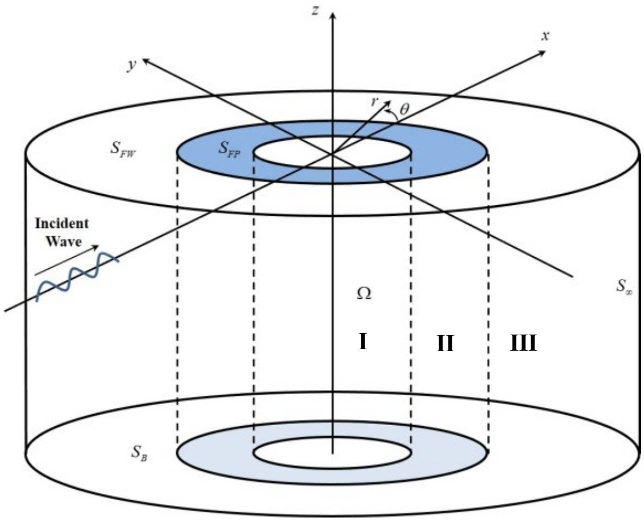

Figure 1. *Basic configuration*

Boundary condition for plate of zero thickness on the free surface

In order to express the boundary condition on the plate-wave interaction region, the kinematic condition and dynamic condition have to be considered. To do this, the governing equation of thin plate deflection is used. The governing equation is given by

$$M\frac{\partial^2 W}{\partial t^2} + D\Delta_0^2 W = P \tag{1}$$

where $W(x, y, t)$ is the deflection, M is the mass of unit area, D is flexural rigidity, P is the external pressure and Δ_0 denotes horizontal Laplace operator expressed as $\Delta_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ to fit with the circular plate. The dynamic condition on the interaction region is that the pressure is equal to hydrodynamic pressure comes from the Bernoulli's equation.

$$M\frac{\partial^2 W}{\partial t^2} + D\Delta_0^2 W = -\rho\frac{\partial \Phi}{\partial t} - \frac{1}{2}\rho\nabla\Phi\cdot\nabla\Phi - \rho g W \qquad on\ z = W(x, y, t) \tag{2}$$

The kinematic condition in this domain states that the normal velocity of the plate is equal to that of water particle.

$$\frac{\partial W}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla\Phi \cdot \nabla W \qquad \text{on } z = W(x, y, \mathrm{t}) \tag{3}$$

These conditions have two difficulties which are nonlinear and are applied at the unknown position $z = W$. In order to transfer the boundary condition from the unknown position to its mean position *z = 0,* the classical way suggested by Stokes is introduced. For the small displacement assumption, we can expand the exact plate deflection to the mean position using the Taylor series expansions.
We can write the dynamic condition:

$$(1 + M\frac{\partial^2}{\partial t^2} + \frac{D}{\rho g}\Delta_0^2)W + \frac{1}{g}\frac{\partial \Phi}{\partial t} = -\frac{1}{2g}\nabla\Phi \cdot \nabla\Phi - \frac{1}{g}W\frac{\partial^2 \Phi}{\partial z \partial t} \qquad on\ z = 0 \tag{4}$$

The kinematic condition can be expressed as following:

$$\frac{\partial W}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla\Phi \cdot \nabla W + W\frac{\partial^2 \Phi}{\partial z^2} + o\left(\Phi^3\right) \tag{5}$$

This expression is written up to the second order with respect to Φ and the notation $o(\varepsilon)$ is used to represent the order higher than ε while $O(\varepsilon)$ denotes the order of ε.
Next step is to seek a solution, Φ and *W*, into a perturbation series with respect to wave steepness $\varepsilon = A / \lambda$ where *A* is the wave amplitude and λ is its length.

$$\Phi = \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + o\left(\varepsilon^3\right) \ , \ W = \varepsilon W + \varepsilon^2 W^{(2)} + o\left(\varepsilon^3\right) \tag{6}$$

For monochromatic incident waves, the time dependency can be separated by using the time periodic assumption at frequency ω . The first- and second order velocity potential have the forms.

$$\phi^{(1)}(x,t) = \Re\left\{\varphi^{(1)}(x)e^{-i\omega t}\right\} \ , \ \phi^{(2)}(x,t) = \overline{\varphi}^{(2)}(x) + \Re\left\{\varphi^{(2)}(x)e^{-2i\omega t}\right\} \tag{7}$$

The similar notations are used for the plate deformation *W.*

$$W^{(1)}(x,y,t) = \Re\left\{w^{(1)}(x,y)e^{-i\omega t}\right\} \ , \ W^{(2)}(x,y,t) = \overline{w}^{(2)}(x,y) + \Re\left\{w^{(2)}(x,y)e^{-2i\omega t}\right\} \tag{8}$$

Because we focused on the high frequency phenomena, the over-bar expression for the steady parts of the second order is neglected in this study.
After inserting these expression into kinematic condition and dynamic condition, we obtain the combined plate-wave interaction boundary condition and plate deflection in frequency domain at corresponding orders.

$$O(\varepsilon): \quad -\nu\varphi^{(1)} + \mathcal{G}^{(1)}\frac{\partial \varphi^{(1)}}{\partial z} = 0,$$

$$w^{(1)} = \frac{i}{\omega}\frac{\partial \varphi^{(1)}}{\partial z};$$

$$O(\varepsilon^2): \ -4\nu\varphi^{(2)} + \mathcal{G}^{(2)}\frac{\partial \varphi^{(2)}}{\partial z} = \frac{i\omega}{g}[\frac{1}{2\nu}\mathcal{G}^{(1)}(\nabla_0\varphi^{(1)} \cdot \nabla_0\frac{\partial \varphi^{(1)}}{\partial z} - \frac{\partial \varphi^{(1)}}{\partial z}\frac{\partial^2 \varphi^{(1)}}{\partial z^2}) + \frac{1}{2}\nabla\varphi^{(1)} \cdot \nabla\varphi^{(1)} + \frac{\partial \varphi^{(1)}}{\partial z}\frac{\partial \varphi^{(1)}}{\partial z}],$$

$$w^{(2)} = \frac{i}{2\omega}\left\{\frac{\partial \varphi^{(2)}}{\partial z} - \frac{1}{2}\nabla_0\varphi^{(1)} \cdot \nabla_0 w^{(1)} + \frac{1}{2}w^{(1)}\frac{\partial^2 \varphi^{(1)}}{\partial z^2}\right\}; \tag{9}$$

where the mathematical operators $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ are defined as follows:

$$\begin{aligned} \mathcal{G}^{(1)} &= (1 - \nu M / \rho + D / (\rho g)\Delta_0^2) \\ \mathcal{G}^{(2)} &= (1 - 4\nu M / \rho + D / (\rho g)\ \Delta_0^2) \end{aligned} \tag{10}$$

Solution methodology

In order to solve the problem, the method of matched eigenfunction expansions is used. The eigenfunction expansion for the plate problem was introduced by Kim & Ertekin [2] and Malenica & Korobkin [4]. We divide the fluid domain into three regions: an outer region defined by $R \le r \le \infty$, $0 \le \theta \le 2\pi$, $-h \le z \le 0$, a middle region (plate-wave interaction) defined by $a \le r \le R$, $0 \le \theta \le 2\pi$, $-h \le z \le 0$ and an inner region defined by $0 \le r \le a$, $0 \le \theta \le 2\pi$, $-h \le z \le 0$. To match the solutions, the continuity of the pressure and normal velocity are introduced at the common boundaries.

Potential decomposition in the inner and the outer regions

The total potential in the inner and outer regions (regions I and III in Figure 1) is divided into the incident and perturbation part. The incident potential, up to second order, is well known and can be written as follows:

$$\varphi_I^{(1)} = -\frac{igA}{\omega}\frac{\cosh k_0(z+h)}{\cosh k_0 h}\sum_{m=0}^{\infty}\varepsilon_m i^m J_m(k_0 r)\cos m\theta \tag{11}$$

$$\varphi_I^{(2)} = -\frac{3i\omega A^2}{8}\frac{\cosh 2k_0(z+h)}{\sinh^4 k_0 h}\sum_{m=0}^{\infty}\varepsilon_m i^m J_m(2k_0 r)\cos m\theta \tag{11}$$

where ε_m is equal to 1 for $m = 0$ and 2 for $m > 0$.

The perturbation potential, at each order *(j)* is now decomposed into two parts $\varphi_{PH}^{(j)}$ and $\varphi_{PQ}^{(j)}$ which satisfy the homogeneous and non-homogeneous free surface boundary conditions respectively:

$$\varphi^{(j)} = \varphi_I^{(j)} + \varphi_P^{(j)} = \varphi_I^{(j)} + \varphi_{PH}^{(j)} + \varphi_{PQ}^{(j)} \tag{13}$$

Since the potential $\varphi_{PH}^{\mathrm{I\ or\ III},(j)}$ satisfies the homogeneous free surface boundary condition ($-\alpha\varphi_{PH}^{(j)} + \partial\varphi_{PH}^{(j)}/\partial z = 0$) can be found in the form:

$$\varphi_{PH}^{III,(j)}(r,\theta,z) = \sum_{m=0}^{\infty}\varepsilon_m\sum_{n=0}^{\infty}A_{mn}^{(j)}f_n^{(j)}(z)H_m^{(1)}(k_n r)\cos m\theta \tag{14}$$

$$\varphi_{PH}^{I,(j)}(r,\theta,z) = \sum_{m=0}^{\infty}\varepsilon_m\sum_{n=0}^{\infty}D_{mn}^{(j)}f_n^{(j)}(z)J_m(k_n r)\cos m\theta \tag{15}$$

where the vertical eigenfunction is $f_n^{(j)}(z) = \cosh k_n(z+h)/\cosh k_n h$ and the eigenvalues k_n are computed by using the dispersion relation, $\alpha = k\tanh kh$ (for the first order: $\alpha = \nu$, second order: $\alpha = 4\nu$). The dispersion relation gives one real root k_0 and infinite number of imaginary roots k_n.

The perturbation potential $\varphi_{PQ}^{\mathrm{I\ or\ III},(j)}$ is chosen to satisfy the homogeneous boundary condition at the vertical boundaries of the domains ($\partial\varphi_{PQ}^{(j)}/\partial r = 0$ at $r = a$ or R) and the non-homogeneous free surface boundary condition ($-\alpha\varphi_{PQ}^{(j)} + \partial\varphi_{PQ}^{(j)}/\partial z = Q_P^{(j)}$). The corresponding solution at the vertical boundaries can be expressed in the following form:

$$\varphi_{PQ}^{III,(j)}(R,\theta,z) = \sum_{m=0}^{\infty}\varepsilon_m\sum_{n=0}^{\infty}L_{mn}^{(j)}f_n^{(j)}(\mathrm{z})\cos m\theta\ ,\quad \varphi_{PQ}^{I,(j)}(a,\theta,z) = \sum_{m=0}^{\infty}\varepsilon_m\sum_{n=0}^{\infty}N_{mn}^{(j)}f_n^{(j)}(\mathrm{z})\cos m\theta \tag{16}$$

where the coefficients L_{mn}, N_{mn} can be deduced by using the Green's theorem as shown in [3].

$$L_{mn} = -\frac{2C_n\int_R^{\infty}H_m^{(1)}(k_n\rho)Q_{Pm}(\rho)\rho d\rho}{k_n R H_m^{(1)'}(k_n R)}\ ,\quad N_{mn} = \frac{2C_n\int_0^{a}J_m(k_n\rho)Q_{Pm}\rho d\rho}{k_n a J_m'(k_n a)} \tag{17}$$

Potential decomposition in the plate region

The potential in the plate region (region II in Figure 1) is also decomposed into two part $\varphi_{PH}^{II,(j)}$ and $\varphi_{PQ}^{II,(j)}$ where the potential $\varphi_{PH}^{II,(j)}$ satisfies the homogeneous plate-wave interaction boundary condition ($-\alpha\varphi_{PH}^{(j)} + \mathcal{G}^{(j)}\partial\varphi_{PH}^{(j)}/\partial z = 0$) can be written in the following form:

$$\varphi_{PH}^{II,(j)} = \sum_{m=0}^{\infty}\varepsilon_m\sum_{n=-2}^{\infty}F_n^{(j)}(B_{mn}^{(j)}J_m(\mu_n r) + C_{mn}^{(j)}Y_m(\mu_n r))\cos m\theta \tag{18}$$

where the vertical eigenfunction is $F_n^{(j)}(z) = \cosh\mu_n(z+h)/\cosh\mu_n h$ and the eigenvalues μ_n are the solution of the dispersion relation, $\alpha = (1-\alpha\mathrm{M}/\rho + \mathrm{D}/(\rho\mathrm{g})\mu^4)\mu\tanh\mu h$). This equation gives one real root (μ_0), infinite number of imaginary roots (μ_n), and two complex roots (μ_{-1} , μ_{-2}). These two complex roots are related to each other as $\mu_{-2} = \mu_{-1}^{*}$ with asterisk denoting the complex conjugate. The complex roots are introduced by Evans & Davies [1].

Finally, the remaining part of the perturbation potential $\varphi_{PQ}^{II,(j)}$, which satisfies the homogeneous boundary condition ($\partial\varphi_{PQ}^{(j)} / \partial r = 0$ at $r=a,\ R$) and the non-homogeneous boundary condition at the plate interface ($-\alpha\varphi_{PQ}^{(j)} + \mathcal{G}^{(j)}\partial\varphi_{PQ}^{(j)} / \partial z = Q_P^{II}$) can be expressed in the form:

$$
\begin{aligned}
\varphi_{PQ}^{II,(j)}(a,\theta,z) &= \sum_{m=0}^{\infty}\varepsilon_m \sum_{n=-2}^{\infty} M_{mn}^{a,(j)} F_n^{(j)}(z)\cos m\theta , \\
\varphi_{PQ}^{II,(j)}(R,\theta,z) &= \sum_{m=0}^{\infty}\varepsilon_m \sum_{n=-2}^{\infty} M_{mn}^{R,(j)} F_n^{(j)}(z)\cos m\theta
\end{aligned}
\tag{19}
$$

where coefficients M_{mn}^{a} , M_{mn}^{R} can be deduced by using the Green's theorem with the following Green's function:

$$
G^{w} = \sum_{m=0}^{\infty}\varepsilon_m \sum_{n=-2}^{\infty} C_m \begin{pmatrix} H_m^{(1)}(\mu_n r) J_m(\mu_n\rho) \\ J_m(\mu_n r) H_m^{(1)}(\mu_n\rho) \end{pmatrix} F_n(z) F_n(\zeta)\cos m(\theta-\vartheta) \quad , \quad \begin{pmatrix} r>\rho \\ r<\rho \end{pmatrix} \tag{20}
$$

Matching and the final linear system of equations for the unknown coefficients

In order to get the unknown coefficients $A_{mn}^{(j)}$, $B_{mn}^{(j)}$, $C_{mn}^{(j)}$ and $D_{mn}^{(j)}$, we need to truncate the infinite series in the expressions for the potentials and after that apply the matching conditions at the vertical boundaries of the different domains:

$$
\left\{\varphi^{II,(j)} = \varphi^{III,(j)}\right\}_{r=R} \quad , \quad \left\{\frac{\partial\varphi^{II,(j)}}{\partial r} = \frac{\partial\varphi^{III,(j)}}{\partial r}\right\}_{r=R} \tag{21}
$$

$$
\left\{\varphi^{II,(j)} = \varphi^{I,(j)}\right\}_{r=a} \quad , \quad \left\{\frac{\partial\varphi^{II,(j)}}{\partial r} = \frac{\partial\varphi^{I,(j)}}{\partial r}\right\}_{r=a} \tag{22}
$$

Finally, in order to properly close the problem we need to apply the plate end boundary conditions. In the case of free ends these conditions should ensure that the bending moment and shear force are zero at the plate ends:

$$
\Delta_0 w^{(j)} - (1-\mathcal{N})\left[\frac{1}{r}\frac{\partial w^{(j)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^{(j)}}{\partial\theta^2}\right] = 0 \quad , \quad \frac{\partial\Delta_0 w^{(j)}}{\partial r} + \frac{1-\mathcal{N}}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2 w^{(j)}}{\partial\theta^2}\right] = 0 \tag{23}
$$

The numerical results for the first- and second order deflection of the plate will be presented at the Workshop.

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