Ship waves at finite depth in the presence of uniform vorticity

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Highlights

- Theory for ship waves in the presence of a shear flow of uniform vorticity is extended to the case of finite water depth.
- The presence of a shear flow, at arbitrary angle with the ship's direction of motion, introduces novel features such as asymmetric wakes, non-constant Kelvin wake angles and critical ship velocity above which transverse waves vanish.
- An explicit expression for the critical velocity with both shear and finite depth is derived, together with limits for the corresponding sector of wave propagation forbidden at supercritical velocities. A subtle interplay between shear flow and water depth is found.

I. INTRODUCTION

The theory of ship waves dates back to Lord Kelvin, who showed in 1887 that the angle formed by the waves in a ship's wake always forms the same angle, $\phi_{\mathrm{K}} = 19°28'$[1]. The theory was developed further, in particular, by Havelock [2, 3], and is reviewed in the classical literature[4, 5]. Ship waves, and the Kelvin angle in particular, have lately received much attention in the literature [6–13]. Recently, the classical theory was extended by one of us to include the presence of a Couette-type shear flow of uniform vorticity below the surface [14], forming an arbitrary angle with the ship's direction of motion.

Several realistic situations involve the presence of shear flow beneath the water surface; examples include shallow rivers, sub-surface currents, and when the water near the surface is set in motion by wind (e.g. [15]). As a model flow we consider the simplest shear flow, namely a flow of uniform vorticity (Couette profile). While somewhat idealised compared to real flows (note that G.I. Taylor observed this kind of profile where a bubble curtain surfaces [16]), it allows reasonably straightforward analysis, and is an important stepping stone towards understanding the interaction of ship waves with more general shear profiles.

The presence of a shear flow below the surface was found in Ref. [14] to have a profound influence on the ship waves, determined mainly by the dimensional group $\mathrm{Fr}_S = VS/g$ where S is the shear (vorticity), V is the speed of the ship and g the acceleration of gravity. It may be interpreted as a "shear Froude number" based on the length g/S^2, and implies, crucially, that even moderate shear becomes important for fast–moving ships.

In the following we extend the theory to the case where the water has a finite depth h. While the re-derivation of the governing equations is only slightly complicated by the incorporation of the extra parameter, the introduction of a finite depth is by no means trivial, since the effects of water depth and shear flow interact with each other.

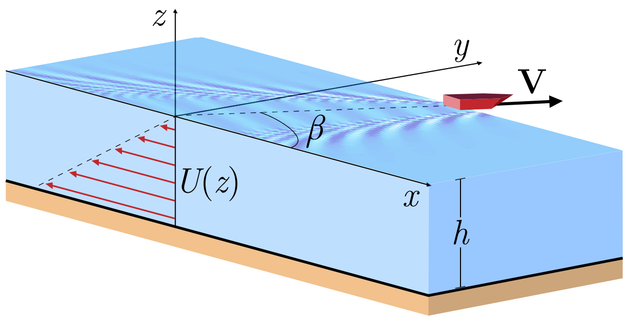

FIG. 1. The geometry considered.

A particularly interesting point reported in Ref. [14] is that the presence of the shear flow implies that in every direction of ship motion except exactly along the shear current, there exists a critical ship velocity V_{crit} beyond which no transverse waves are produced by the boat. The physical reason is that the shear limits the phase velocity of even very long waves, and that transverse waves, propagating in directions close to that of the ship's movement, cannot keep up with a ship moving faster than the maximum phase velocity. Exactly the same phenomenon is well known to appear in the case of ship waves at finite depth [3], where the phase velocity can never exceed $c_{\max} = \sqrt{gh}$.

II. THEORY

The geometry of the problem is shown in Fig. 1. A ship travels at velocity $\mathbf{V}$ relative to the surface of the water. We choose the coordinate system so that the surface velocity is zero, and the basic flow is $U(z) = Sz$ along the x axis. The water depth is h, the water density is ρ and we assume incompressible flow.

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We let β be the angle between the ship's motion and the shear flow as shown in Fig. 1. The ship perturbs the basic flow so that the velocity and pressure field become

$$\mathbf{v} = [U(z) + \hat{u}, \hat{v}, \hat{w}]; \quad p = -\rho g z + \hat{p}. \tag{1}$$

We work to linear order in these perturbations. Noting that ship waves must appear stationary as seen from the moving ship, the perturbed quantities can depend on time only through the combination $\boldsymbol{\xi} = \mathbf{r}_\perp - \mathbf{V}t$ where $\mathbf{r}_\perp = (x, y)$. We therefore subject all physical quantities to the plane Fourier transform

$$\hat{u}(\boldsymbol{\xi}, z) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} u(\mathbf{k}, z) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\xi}}, \tag{2}$$

etc. Here $\mathbf{k} = (k_x, k_y) = (k\cos\theta, k\sin\theta)$. The physical value is the real part.

Because of the presence of vorticity in an essentially three–dimensional problem, the velocity potential could not be used, and it was necessary to solve the full Euler equations, where the only assumptions made was incompressibility and that viscosity may be neglected. The Euler equations and continuity equation then read

$$ik_x u + ik_y v + w' = 0; \tag{3a}$$
$$i(k_x U - \mathbf{k}\cdot\mathbf{V})u + Sw = -ik_x p/\rho; \tag{3b}$$
$$i(k_x U - \mathbf{k}\cdot\mathbf{V})v = -ik_y p/\rho; \tag{3c}$$
$$i(k_x U - \mathbf{k}\cdot\mathbf{V})w = -p'/\rho, \tag{3d}$$

where a prime denotes derivation w.r.t. z. We can eliminate u, v and p from these equations to obtain an equation for w alone (Rayleigh equation): $w'' = k^2 w$. When subjected to the boundary condition $w(k, -h) = 0$ we obtain the solutions

$$u = iA\left[k_x \cosh k(z+h) + \frac{S k_y^2 \sinh k(z+h)}{k(k_x U - \mathbf{k}\cdot\mathbf{V})}\right] \tag{4a}$$
$$v = iA\left[k_y \cosh k(z+h) - \frac{S k_x k_y \sinh k(z+h)}{k(k_x U - \mathbf{k}\cdot\mathbf{V})}\right] \tag{4b}$$
$$w = kA \sinh k(z+h) \tag{4c}$$
$$p = -iA[(k_x U - \mathbf{k}\cdot\mathbf{V})\cosh k(z+h) - S\cos\theta \sinh k(z+h)]. \tag{4d}$$

where $A(\mathbf{k})$ is as yet undetermined.

The ship is modelled as a travelling pressure disturbance on the surface, chosen to have Gaussian form for ease of comparison with literature [7, 8]:

$$\hat{p}_{\mathrm{ext}} = p_0 \mathrm{e}^{-\pi^2 \xi^2/b^2}; \quad p_{\mathrm{ext}} = (b^2 p_0/\pi)\mathrm{e}^{-k^2 b^2/(2\pi)^2} \tag{5}$$

where p_{ext} is the Fourier transform of $\hat{p}_{\mathrm{ext}}$ and $\xi = |\boldsymbol{\xi}|$. Here b is the "size" of the ship, and the ship's Froude number is $\mathrm{Fr}_b = V/\sqrt{gb}$. Moreover we define the surface elevation $\zeta(\boldsymbol{\xi})$ relative to an undisturbed surface to be

$$\zeta(\xi) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} B(\mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\xi}}. \tag{6}$$

Inserting the above solutions and definitions into the dynamic and kinematic boundary conditions at the free surface now gives two equations with A and B as unknown, from which we eliminate A to produce

$$B(\mathbf{k}) = -\frac{1}{\rho}\frac{k p_{\mathrm{ext}}(k)}{gk - (\mathbf{k}\cdot\mathbf{V})^2 \coth kh - S\cos\theta(\mathbf{k}\cdot\mathbf{V})}. \tag{7}$$

This, in principle, is the full solution for the surface wave, where the generalisation from the infinite depth case reported in Ref. [14] is only the extra factor $\coth kh$ in the denominator. While seemingly innocuous, the added factor not only complicates the further analysis but its physical repercussions are also profound.

III. THE FAR-FIELD SOLUTION

The linear theory of ship waves, indeed of any waves generated by a localised source, encounter the same difficulty, that resulting expressions for the surface elevation contain a pole (or several) in the $\mathbf{k}$ plane. As discussed at length in the classical literature (e.g. [5, 17], the contribution from this pole contains the far-field, while further contributions to the integral vanish as $\xi \to \infty$. Physically this may be understood by noting that the zero of the denominator of (7) corresponds exactly to the condition

$$V\cos(\theta - \beta) = c(\mathbf{k}) \tag{8}$$

where [18]

$$c(\mathbf{k}) = \sqrt{(g/k)\tanh kh + (S/2k)^2 \cos^2\theta \tanh^2 kh} - (S/2k)\cos\theta \tanh kh \tag{9}$$

is the phase velocity for a wave vector $\mathbf{k}$. Equation (8) results from insisting that waves seen in the far–field must satisfy the dispersion relation and have a phase velocity which produces waves which are stationary as seen from the source. In the following it will be convenient to work with the angle $\gamma = \theta - \beta$ instead of θ.

A complication is now that unlike the case of infinite depth, no explicit expression for the value of k at the pole may be found. The solution $K(\gamma)$ must instead be found numerically. The contribution from the pole is found by using the Sokhotsky-Plemelj theorem exactly as in Ref. [14], and the resulting expression in the far–field may be written

$$\zeta(\boldsymbol{\xi}) = -\frac{1}{4\rho\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}\gamma K(\gamma) p_{\mathrm{ext}}(K(\gamma)) \times \frac{\sin[K(\gamma)\xi\cos(\gamma + \beta - \phi)]}{\frac{\partial}{\partial k}G(k,\gamma)|_{k=K(\gamma)}}; \tag{10}$$

$$G(k,\gamma) = k - \frac{g}{V^2}\left[\frac{1}{\cos^2\gamma} - \frac{\mathrm{Fr}_S \cos(\gamma+\beta)}{\cos\gamma}\right]\tanh kh, \tag{11}$$

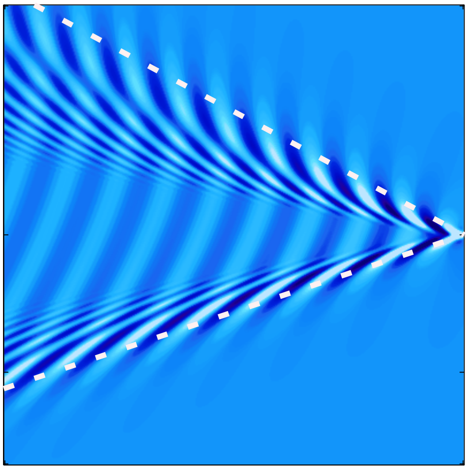

FIG. 2. Example of an asymmetric ship wake resulting from a sub–surface shear current normal to the ship's direction of motion. Here $\beta = \pi/2, \mathrm{Fr}_b = 0.8, \mathrm{Fr}_S = 0.5$ and $h = \infty$. The ship travels towards the right, and the dashed lines show the Kelvin angles (as defined in Ref. [14]) on either side of the wake.

where we have noted that only angles $|\gamma| < \pi/2$ can possibly satisfy Eq. (8), i.e., the wave propagation direction must have positive component along the direction of ship motion. $K(\gamma)$ now solves $G(k, \gamma) = 0$. An example of a boat wake with side–on shear is shown in figure 2, where the asymmetry both of the wake and of the Kelvin angles is clear to see.

IV. THE CRITICAL VELOCITY

When the velocity exceeds a critical value, it is necessary to further restrict the integration sector in Eq. (10), because waves propagating close to parallel with the ship motion (so–called transverse waves) are unable to keep up. To wit, equation (8) cannot be satisfied in a propagation direction θ for which $V\cos\gamma > c_{\max}(\theta)$ where $c_{\max}$ is the maximum phase velocity in said direction. From Eq. (9) one quickly verifies that the maximum velocity is found as $k \to 0$, hence the velocity V is supercritical when,

$$V\cos\gamma > \sqrt{gh + [\tfrac{1}{2}Sh\cos(\gamma+\beta)]^2} - \tfrac{1}{2}Sh\cos(\gamma+\beta).$$

We can re-write the condition for supercriticality as

$$\max_{\gamma}[\mathrm{Fr}_S\cos\gamma\cos(\gamma+\beta) + \mathrm{Fr}_h\cos^2\gamma] > 1 \tag{12}$$

where $\mathrm{Fr}_h = V/\sqrt{gh}$ and the maximum is taken with respect to γ. Some straightforward but tedious algebra reveals that the maximum of the left hand side of (12) is found where $\tan\gamma = -\frac{1}{2}\mathrm{Fr}_S\sin\beta$. This is the propagation direction whose waves are first to vanish once the velocity exceeds the critical. Inserting this back into (12) we may write the condition for the velocity to be supercritical to be

$$\mathrm{Fr}_S(\cos\beta + \tfrac{1}{2}\mathrm{Fr}_S\sin^2\beta) + \mathrm{Fr}_h^2 > 1. \tag{13}$$

At infinite depth ($\mathrm{Fr}_h \to 0$) the critical value of Fr_S was found in Ref. [14] to be $\mathrm{Fr}_{S,\mathrm{crit}} = 1/\cos^2(\beta/2)$. That what we have found is a generalisation of this is obvious when noting that Eq. (13) may be written instead as

$$\frac{\mathrm{Fr}_h^2}{1 + \mathrm{Fr}_S\sin^2(\beta/2)} + \mathrm{Fr}_S\cos^2(\beta/2) > 1. \tag{14}$$

Solved with respect to velocity, the critical value of V can be found as

$$V_{\mathrm{crit}} = 2\sqrt{gh}\frac{\sqrt{\mathrm{Fr}_{Sh}^2 + 4} - \mathrm{Fr}_{Sh}\cos\beta}{4 + \mathrm{Fr}_{Sh}^2\sin^2\beta} \tag{15}$$

where we defined the dimensionless group $\mathrm{Fr}_{Sh} = S\sqrt{h/g}$ (again a Froude number, with respect to length h and velocity Sh). Eq. (15) is readily found to have the appropriate limits as $h \to \infty$ and $S \to 0$.

We may finally derive, from Eq. (12), the angular sector in which waves are unable to keep up with the source and must be excluded from the integral of Eq. (10). A little algebra gives the two cutoff angles as

$$\gamma_{\mathrm{co}}^{\pm} = \arctan\Big[-\tfrac{1}{2}\mathrm{Fr}_S\sin\beta \pm \sqrt{(\tfrac{1}{2}\mathrm{Fr}_S\sin\beta)^2 + \mathrm{Fr}_h^2 + \mathrm{Fr}_S\cos\beta - 1}\Big].3g \tag{16}$$

Angles $\gamma_{\mathrm{co}}^- < \gamma < \gamma_{\mathrm{co}}^+$ must be excluded from the integral, and only waves travelling at a sufficiently large angle with the ship's motion (diverging waves) may contribute to the wake.

V. DISCUSSION

The wake of waves behind a moving ship can change quite radically when a shear flow is present beneath the surface. The wake can be asymmetric and have a smaller or larger Kelvin angles than for uniformly flowing water, depending on the relative direction of motion of ship and shear flow. The presence of shear also limits the phase velocity of waves, introducing a critical velocity beyond which the transverse part of the wake disappears. This phenomenon is previously known from ship waves on water of finite depth. In the present endeavour we have laid out the theory for ship waves in the presence of both shear flow and finite depth, and found the critical ship velocity as an explicit function of shear, depth and the ship's direction of motion.

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