Dispersion relation and instability onset of Faraday waves

DIDIER CLAMOND AND JEAN RAJCHENBACH

Université de Nice – Sophia Antipolis, Parc Valrose, 06108 Nice cedex 2, France.
E-Mail: didier.clamond@gmail.com, jean.rajchenbach@unice.fr

Highlights: Faraday waves; parametric excitation; dispersion relation; stability; bifurcation; wavenumber selection.

1 Introduction

Many studies have been devoted to the phenomenon of Faraday waves, which appear at the free surface of a fluid when the container is submitted to periodical vertical oscillations [3, 9, 12]. The interest of this setup is that it gives rise to the formation of various patterns. According to the forcing amplitude, frequency and fluid viscosity, the free surface can exhibit standing solitary waves [2, 16, 19] or patterns of different symmetry, such as stripes, squares, hexagons, quasicrystalline ordering, or star-shaped waves [4, 5, 6, 15]. These symmetry breaking result from the nonlinear couplings between waves. Thus, the study of Faraday waves constitutes a privileged way to explore complex nonlinear phenomena by the mean of a simple experimental device. Understanding these waves has also applications in hydrodynamics, for instance in sloshing related problems.

Despite noticeable advances in the theoretical understanding of Faraday waves [11, 13, 14, 18] some of their fundamental properties remain into darkness. For instance, the relation of dispersion $\omega(k)$ of parametrically-forced water waves is often erroneously identified with that of free, unforced surface waves; this approximation holding only without forcing and without dissipation. However, the knowledge of the exact dispersion relation is of crucial importance.

The first aim of this work is to establish the actual relation of dispersion of Faraday waves for *nonzero* forcing and dissipation. As shown below, the dispersion relation of free, unforced waves is significantly altered in the case of parametrically–forced excitations: two different wavenumbers correspond then to the same angular frequency. We carry out their stability analysis and we discuss the nature of the bifurcation giving rise to the wavy surface state from the rest state when the forcing is increased. Thus, the threshold of the Faraday instability is established as well as the selected wavenumbers in both cases of short and long waves. At last, it is shown that the transition can be either smooth (supercritical) or discontinuous and hysteretic (subcritical), depending on the thickness of the liquid layer.

2 Mathieu equation

Consider a container partly filled with a Newtonian fluid of depth d, moving up and down in a purely sinusoidal motion of angular frequency Ω and amplitude $\mathcal{A}$, so that the forcing acceleration is $\Omega^2\mathcal{A}\cos(\Omega t)$. In the reference frame moving with the vessel, the fluid experiences a vertical acceleration due to the apparent gravity $G(t) \equiv g - \Omega^2\mathcal{A}\cos(\Omega t)$, g being the gravity acceleration in the laboratory frame of reference and t being the time.

Let be $\boldsymbol{x} = (x_1, x_2)$ and y respectively the horizontal and upward vertical Cartesian coordinates moving with the vessel. Ordinates $y = -d$, $y = 0$ and $y = \eta(\boldsymbol{x}, t)$ respectively correspond to the horizontal impermeable bottom, of the liquid level at rest and of the impermeable free surface. The Fourier transform of the latter is $\zeta(\boldsymbol{k}, t) \equiv \iint_{-\infty}^{\infty} \eta(\boldsymbol{x}, t) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}^2\boldsymbol{x}$, where $\mathrm{i}^2 = -1$ and $\boldsymbol{k}$ is the wave vector with $k = |\boldsymbol{k}|$.

For parametrically-driven infinitesimal surface waves, ζ is described by a damped Mathieu equation [3, 4]

$$\zeta_{tt} + 2\,\sigma\,\zeta_t + \omega_0^2\,[\,1\,-\,F\cos(\Omega t)\,]\,\zeta \;=\; 0, \quad (2.1)$$

where $\sigma = \sigma(k)$ is the viscous attenuation, $\omega_0 = \omega_0(k)$ is the angular frequency of linear waves without damping and without forcing, and $F = F(k)$ is a dimensionless forcing. For pure gravity waves in finite depth, we have

$$\omega_0^2 \;=\; gk\,\tanh(kd), \qquad F \;=\; \Omega^2\,\mathcal{A}\,/\,g. \qquad (2.2)$$

In (2.1), the damping coefficient σ originates in the bulk viscous dissipation and in the viscous friction with the bottom in the case of shallow water. For free gravity waves in the limit of small viscosity, we have [7, 8]

$$\sigma\ = \nu k^2 \left[2 + \frac{\coth(2kd)}{\sinh(2kd)}\right] + \frac{k\sqrt{k\nu c_0/2}}{\sinh(2kd)} \quad (2.3)$$

where ν is the fluid kinematic viscosity and $c_0 = \sqrt{gd}$. The first term in the right-hand side of (2.3) represents the bulk dissipation, while the second one models the friction with the bottom.

It is well known that systems obeying a (damped) Mathieu equation with excitation angular frequency Ω exhibit a series of resonance conditions for response angular frequencies ω equal to $n\Omega/2$, n being an integer [1]. These solutions are expressed in term of the Mathieu functions together with a dispersion relation involving the so-called *Floquet exponent*. Mathieu functions are transcendent and cannot be expressed in term of simple functions in closed form. In order to understand qualitatively these solution we therefore consider here approximations in the limit of small forcing and dissipation.

Assuming $F \ll 1$ and $\sigma \sim \mathrm{O}(F)$, an approximate dispersion for the sub-harmonic response ($n = 1$) is (with $n = 1$, $\omega = \Omega/2$)

$$\omega_0 \,/\, \omega \ \approx\ 1 \ \pm\ \sqrt{(F/4)^2 - (\sigma/\omega)^2}, \qquad (2.4)$$

where ω_0 is related to k via (2.2). One condition to obtain stationary waves is that ω_0 is real, thus defining a threshold $F_\downarrow = 4\sigma/\omega$ with $F > F_\downarrow$ for the forcing in order to obtain time-periodic waves. Interestingly, we note that there are *two* wavenumbers k corresponding to the same wave angular frequency ω (for Ω, F and σ given), whatever the relation $\omega_0 = \omega_0(k)$.

Assuming now $F \ll 1$ and $\sigma \sim \mathrm{O}(F^2)$, an approximate dispersion for the harmonic response ($n = 2$, $\omega = \Omega$) is

$$\omega_0 \,/\, \omega \ \approx\ 1 \ +\ \tfrac{1}{12}F^2 \ \pm\ \sqrt{\tfrac{1}{64}F^4 - (\sigma/\omega)^2}. \quad (2.5)$$

The condition of reality for ω defines the threshold $F^2 \geqslant 8\sigma/\omega$. Analog approximations for all n can be easily derived.

Despite a limited range of validity, these relations clearly demonstrate that two wavenumbers (i.e., two $\omega_0 \equiv \omega_0^\pm$) correspond to the angular frequency $\omega = n\Omega/2$. Equations (2.4) and (2.5) result from a linear approximation, and their validity is restricted to waves of infinitesimal amplitude. However, nonlinearities play a significant role for waves of finite amplitudes, and we will now look closely at the nonlinear effects in an amplitude equation.

3 Amplitude equation

Seeking for an approximation in the form $\eta(x,t) = \mathrm{Re}\{A(t)\}\cos(kx) + \mathrm{O}\big(A^2\big)$, assuming $|kA| \ll 1$ and weak forcing and dissipation (i.e., $F \sim \mathrm{O}(|A|^2)$ and $\sigma \sim \mathrm{O}(|A|^2)$) an equation for the slowly modulated amplitude A can be derived in the form [10, 13]

$$\frac{\mathrm{d}A}{\mathrm{d}t} \ +\ (\sigma - \mathrm{i}\omega_0)\,A \ -\ \frac{F\Omega}{8\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\Omega t}A^* \ -\ \frac{K\Omega k^2}{2\mathrm{i}}\,|A|^2A \ =\ 0, \qquad (3.1)$$

a star denoting the complex conjugate. It is obvious that the sign of the nonlinear term in (3.1), via the sign of K, plays a key role in the stability of the solutions.

For pure gravity waves on finite depth, we have [17] (with $s = \mathrm{sech}(2kd)$)

$$K \ =\ \frac{2 - 6s - 9s^2 - 5s^3}{16(1+s)(1-s)^2}.$$

Note that K changes sign with the depth: $K > 0$ for short waves, $K < 0$ for long waves and $K = 0$ for $kd \approx 1.058$. Defining $B = A\exp(\frac{\mathrm{i}}{4}\pi - \frac{\mathrm{i}}{2}\Omega t)$, (3.1) yields

$$\frac{\mathrm{d}B}{\mathrm{d}t} \ =\ \left(\mathrm{i}\omega_0 + \frac{\Omega}{2\mathrm{i}} - \sigma\right)B \ +\ \frac{F\Omega}{8}B^* \ +\ \frac{K\Omega k^2}{2\mathrm{i}}|B|^2B, \quad (3.2)$$

which is a more convenient form for the analysis below.

We focus now on two solutions of (3.2) that are of special interest here: the rest $B = 0$ and the standing wave of constant amplitude. The first one is trivial and we investigate below its stability. The second one is obtained seeking for solutions of the form $B = a\exp(\frac{\mathrm{i}}{4}\pi - \mathrm{i}\delta)$, a and δ being constants, equation (3.2) yielding thus

$$\frac{\omega_0}{\omega} \ =\ 1 \ +\ K(ka)^2 \ \pm\ \sqrt{\frac{F^2}{16} - \frac{\sigma^2}{\omega^2}}, \quad (3.3)$$

with $\omega = \Omega/2$. As $a \to 0$, the approximate dispersion relation (2.4) is recovered. If $F = \sigma = 0$, the dispersion relation of weakly nonlinear, unforced, standing waves in finite depth is recognised too. Therefore, compared to free nonlinear waves, the dispersion relation of parametrically-forced waves is characterised by the shift in angular frequency $\Delta\omega = \pm\sqrt{(F\omega/4)^2 - \sigma^2}$ independent of the wave amplitude a.

In the subsequent discussion, we consider that the relation $K(k)$ is uni-valued and we limit our study to the case $K > 0$ (for $K < 0$ the analysis is similar replacing $\omega_0 - \omega$ by $\omega - \omega_0$).

According to the equation (3.3), we have

$$K(ka)^2 = \frac{\omega_0}{\omega} - 1 \mp \sqrt{\left(\frac{F}{4}\right)^2 - \left(\frac{\sigma}{\omega}\right)^2}, \quad (3.4)$$

with the constraint $K(ka)^2$ to be real and positive. The last term in the right-hand side of (3.4) being real, the forcing F must exceed a minimum value $F_\downarrow = 4\sigma/\omega$ to generate at least a stationary nonzero amplitude wave, as already mentioned above. The condition $F > F_\downarrow$ being fulfilled, if we have moreover $F < F_\uparrow$ with

$$F_\uparrow \equiv 4\,\omega^{-1}\sqrt{(\omega_0 - \omega)^2 + \sigma^2},$$

there are two stationary solutions of nonzero amplitude of the dispersion relation (in addition to the solutions with the opposite phase and to the rest solution $B = 0$).

If $F > F_\uparrow$, the positivity of the right-hand side of (3.4) yields only one nonzero solution of (3.1) (in addition to the solution with the opposite phase and to the rest solution). Thus, disregarding the phase, the flat surface is the unique solution for $F < F_\downarrow$, there are three solutions (one being the rest) in the range $F_\downarrow < F < F_\uparrow$, and two solutions (one being the rest)) in the range $F > F_\uparrow$. An important question to address now is, whether or not, these stationary solutions are stable.

4 Stability analysis

Introducing a small perturbation into the stationary solutions of the amplitude equation (3.2), we look for the eigenvalues of the linearised system of equations obeyed by the perturbation. The stability analysis that we conduct below resembles that carried out in [10] for the parametric pendulum. However, a *major* difference is that the eigenfrequency of a freely-oscillating pendulum is unique, whereas free, unforced, water waves exhibit a continuous spectrum of mode frequencies. Moreover, the sign of the nonlinear terms in the wave equation depend on the depth [17].

First, we study the bifurcation from rest (i.e., the stability of the trivial solution $B = 0$). The linearised equation (3.2) has two eigenvalues λ_1 and λ_2 such that

$$\lambda_j = -\sigma + (-1)^j \sqrt{(F\,\omega\,/\,4)^2 - (\omega - \omega_0)^2}.$$

If $(F\omega/4)^2 < (\omega - \omega_0)^2 + \sigma^2$, the real parts of both eigenvalues are negative. Therefore, the rest is stable. If $(F\omega/4)^2 > (\omega - \omega_0)^2 + \sigma^2$, the eigenvalue λ_2 is real and positive. Therefore, the rest is unstable and

$$F_\uparrow = 4\sqrt{(1 - \omega_0/\omega)^2 + (\sigma/\omega)^2} \qquad (4.1)$$

corresponds to the minimal forcing necessary to destabilise the rest state and to generate surface waves.

Second, we analyse the stability of the permanent solutions of finite amplitude $a > 0$ of the amplitude equation (3.2). We consider, for simplicity, small perturbations in the form $B = [a + b(t)] \exp \mathrm{i}(\pi/4 - \delta)$, a, δ and ω_0 being given in (3.3), and b a complex amplitude to be determined such that $|b| \ll a$. To the linear approximation, the eigenvalues of the resulting equation are ($j = 1, 2$)

$$\lambda_j = -\sigma + (-1)^j \times \sqrt{\sigma^2 - K(2\omega ka)^2\left[1 - \omega_0^{\pm}/\omega + K(ka)^2\right]}.$$

The criterion for having both eigenvalues real and negative is $1 - \omega_0^{\pm}/\omega + K(ka)^2 > 0$. This inequality is to be coupled with (3.4). Thus, it appears clearly that, for the case $K > 0$, the two eigenvalues are both negative if $\omega_0 = \omega_0^-$, thence $\omega_0 < \omega$. The corresponding stationary solution is therefore stable. The other stationary solution $\omega_0 = \omega_0^+$, existing in the range $F_\downarrow < F < F_\uparrow$, corresponds to $\lambda_1 < 0$ and $\lambda_2 > 0$ and is therefore unstable.

Note that the neutrally stable limiting case $\lambda_2 = 0$ is obtained for $F = F_\downarrow$ or $ka = 0$ or $K = 0$. The two first cases correspond to the rest (i.e., no waves), while the third one requires a higher-order equation to conclude on the stability. Note also that the opposite conclusions hold for $K < 0$: the stable solution corresponds then to $\omega_0 = \omega_0^+$ (i.e., $\omega_0 > \omega$).

5 Wavenumber selection

We can now determine the wavenumbers selected at the instability onset. The minimal forcing required to destabilise the free surface from rest is given by (4.1), where ω_0 is related to the wavenumber k by (2.2), the dissipation factor σ being given in (2.3). The first wave to emerge from rest is the one requiring the smaller value of $F_\uparrow$, i.e., this wave corresponds to the wavenumber such that $\partial F_\uparrow / \partial k = 0$, i.e.,

$$\frac{\partial F_\uparrow}{\partial k} \;=\; \frac{16\,(\omega_0 - \omega)}{\omega^2\, F_\uparrow}\,\frac{\partial\, \omega_0}{\partial k} \;+\; \frac{16\,\sigma}{\omega^2\, F_\uparrow}\,\frac{\partial\, \sigma}{\partial k} \;=\; 0, \tag{5.1}$$

together with $\omega = \Omega/2$.

In the limiting case of deep water (i.e., $d = \infty$, $\omega_0 = \sqrt{gk}$, $\sigma = 2\nu k^2$, the most unstable wavenumber k given by (5.1) is

$$2\omega_0 \;=\; \omega \,+\, \sqrt{\omega^2 - 16\sigma^2}.$$

In the opposite limit of long waves (i.e. $kd \ll 1$, $\omega_0 = k\sqrt{gd}$, $\sigma = (gd)^{1/4}\sqrt{k\nu/8d^2}$), the most unstable wavenumber corresponds to

$$\omega_0 \;=\; \omega - 16\nu/d^2.$$

In both cases, the first mode emerging from the rest is such that $\omega_0 < \omega$. The same conclusion arises for arbitrary depth and with surface tension under quite general assumptions (to be explained at the conference). We conclude that the critical mode $\omega_0(k)$ selected at the destabilisation threshold $F_\uparrow$ of the rest state fulfils the inequality $\omega_0 < \omega = \Omega/2$ in both cases of short and long waves. However, as we mentioned above, the sign of the nonlinear term in equation (3.1) depends on the depth, K being positive for short waves, and negative for long ones. Therefore, we conclude that the transition from the rest to the wavy state is *subcritical* (i.e., with hysteresis) for long waves, and *supercritical* (i.e., smooth) for short waves.

6 Conclusion

The dispersion relation of Faraday waves is modified compared to that of free, unforced waves: the forcing amplitude and the dissipation play a key role in the relation of dispersion. For a given forcing frequency, there are two corresponding wavenumbers. We have determined the value of the forcing at the instability threshold, in both cases of short and long waves, as well as the selected wavenumbers. We have also studied the nature of the bifurcation at the instability onset, and we have shown that the transition is supercritical for short waves and subcritical for long waves.

References

[1] M. Abramowitz, M. & I. A. Stegun, *Handbook of Mathematical Functions*. Dover, (1965).

[2] H. Arbell & J. Fineberg, Phys. Rev. Lett. **85**, 756 (2000).

[3] T. B. Benjamin & F. Ursell, Proc. R. Soc. Lond. A **225**, 505 (1954)

[4] S. Ciliberto & J.P. Gollub, J. Fluid Mech. **158**, 381 (1985).

[5] B. Christiansen, P. Alstrøm & M. Levinsen, Phys. Rev. Lett.**68**, 2157 (1992).

[6] S. Douady & S. Fauve, Europhys. Lett. **6**, 221 (1988).

[7] S. S. Hough, Proc. Lond. Math. Soc. **28**, 264 (1896).

[8] J. N. Hunt, La Houille Blanche **6**, 685 (1964).

[9] M. Faraday, Phil. Trans. R. Soc. London **52**, 299 (1831).

[10] S. Fauve, in Dynamics of Nonlinear and Disordered Systems, edited by G. Martinez-Meckler and Seligman (World Scientific, Singapore, 1995).

[11] K. Kumar, Proc. R. Soc. Lond. A **452**, 1113 (1996).

[12] J. W. Miles, & D. Henderson, Ann. Rev. Fluid Mech. **22**, 143 (1990).

[13] S. T. Milner, J. Fluid Mech. **225**, 81 (1992).

[14] H. W. Müller, H. Wittmer, C. Wagner, J. Albers & K. Knorr, Phys. Rev. Lett. **78**, pp. 2357 (1997).

[15] J. Rajchenbach, D. Clamond & A. Leroux, Phys. Rev. Lett. **110**, 094502 (2013).

[16] J. Rajchenbach, A. Leroux & D. Clamond, Phys. Rev. Lett. **107**, 024502 (2011).

[17] I. Tadjbakhsh, & J. B. Keller, J. Fluid Mech. **3**, 442 (1960).

[18] K. Kumar, & L. S. Tuckerman, J. Fluid. Mech. **279**, 49 (1994).

[19] J. Wu, R. Keolian & I. Rudnick, Phys. Rev. Lett. **52**, 1421 (1984)