Two-dimensional breaking wave impact on a vertical wall

by

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1 Introduction
We investigate the violent slamming of a steep wave onto a vertical wall. The novelty of our work relies on the assumption of a breaking type impact. With reference to the plane impermeable wall of Fig. 1, it is assumed that a thin air pocket is formed adjacent to the wall, between the lower and the upper impacted sections of the wall. The solution process, as outlined in the sequel, could be extended to accommodate an uneven bottom configuration. However, in the present study it is explicitly assumed that the bottom is horizontal.

![Definition sketch](attachment:image.png)

**Figure 1** Definition sketch; The curves are free surfaces on which $\phi = 0$.

The boundary conditions at $x=0$ in the intervals $0 < z < \alpha$ and $\beta < z < h$ are defined by default through the normal components of the velocities of the wave before impact, which are assumed to be the constants $V_1$ and $V_2$, respectively. In the general case we assume that $V_1 \neq V_2$. In the intermediate region $\alpha < z < \beta$ the associated boundary condition is governed by the pressure which can be a function of $z$. The present analysis assumes that the pressure impulse (and accordingly the potential) in the air pocket is zero. Nevertheless the outlined methodology can be extended to accommodate non zero constant potential using the same procedure. It is also assumed that at the time of impact the width of the intermediate section between the wave front and the wall $\varepsilon \to 0$.

2 The mixed boundary value problem
In the realm of potential theory, the mixed boundary value problem in terms of $\phi(x,z)$, which is the sudden change in the velocity potential is:

\begin{align}
\nabla^2 \phi &= 0, & (x \to 0, & \ 0 < z < h), \\
\partial \phi / \partial z &= 0, & (x > 0, & \ z = 0), \\
\phi &= 0, & (x > 0, & \ z = h), \\
\partial \phi / \partial z &= V_1, & (x = 0, & \ 0 < z < \alpha), \\
\phi &= 0, & (x = 0, & \ \alpha < z < \beta), \\
\partial \phi / \partial z &= V_2, & (x = 0, & \ \beta < z < h), \\
\phi &\to 0, & (x \to \infty, & \ 0 < z < h).
\end{align}

The form of the solution that satisfies eqs. (1)-(3) and the far-field behaviour (7) is
\[ \phi(x, z) = \sum_{n=1}^{\infty} C_n \cos(\lambda_n z) e^{-\lambda_n x} \]  

where \( \lambda_n = (n - 1/2)\pi/h \), \( n \in \mathbb{N} \) and \( C_n \) are coefficients to be found from the remaining boundary conditions (4)-(6). Note that the present analysis is applied at the instant of the impact and hence \( C_n \) are constants, independent of time.

### 3 Triple trigonometrical series

Introducing eq. (8) into the boundary conditions (4)-(6) we find the following problem that involves triple trigonometrical series

\[ \sum_{n=1}^{\infty} \lambda_n C_n \cos(\lambda_n z) = -V_1, \quad (0 < z < \alpha), \quad \alpha > 0 \]  

\[ \sum_{n=1}^{\infty} C_n \cos(\lambda_n z) = 0, \quad (\alpha < z < \beta), \quad \beta > 0 \]  

\[ \sum_{n=1}^{\infty} \lambda_n C_n \cos(\lambda_n z) = -V_2, \quad (\beta < z < h). \]  

Most literature on mixed boundary value problems that involve trigonometrical series, concern dual – not triple – relations. Relevant examples are the studies of Trantner [1-3]. For a review of dual trigonometrical series the reader is referred to the classical book of Sneddon [4], which cites nearly all studies prior to the time it was published. Triple trigonometrical series concern more complicated mixed-boundary value problems and relevant examples are the works of Trantner [5] and Kerr et al. [6]. However, in both papers the boundary data in two of the three boundary sets are identically zero.

The required analysis to solve the mixed problem governed by eqs. (9)-(11) is quite complicated and therefore only the basic steps are given in the sequel. After several mathematical manipulations, the triple series of eqs. (9)-(11) are transformed into the following three conditions on new coefficients \( B_n \)

\[ \sum_{n=1}^{\infty} B_n J_{\nu}(\mu_n y) = \sqrt{2/(\pi y)} F(\beta y), \quad (0 < y < b), \quad \nu = -1/2, \quad \beta = \alpha / \beta, \quad \mu_n = \lambda_n \beta, \quad y = z / \beta. \]  

The next step is to reduce the triple trigonometrical series to dual by satisfying the last one (14). This is accomplished assuming an alternative form of the unknown constants \( B_n \) according to which

\[ B_n = \frac{1}{\mu_n^{1+p/2} J_{\nu+1}(\mu_n \alpha)} \sum_{m=0}^{\infty} E_m J_{\nu+2m+1+p/2}(\mu_n \alpha), \quad -1 \leq p \leq 1. \]  

Eq. (15) satisfies eq. (14) as it holds that [7]

\[ \sum_{n=1}^{\infty} J_{\nu+2m+1+p/2}(\mu_n \beta) J_{\nu}(\mu_n y) = 0, \quad y > 1. \]  

Introducing eq. (15) into eqs. (12) and (13) the problem is reduced to the dual trigonometrical series

\[ E_0 a_1(y) + \sum_{m=1}^{\infty} E_m \sum_{n=1}^{\infty} \frac{J_{\nu+2m+1+p/2}(\mu_n \beta) J_{\nu}(\mu_n y)}{\mu_n^{1+p/2} J_{\nu+1}(\mu_n \beta)} = \sqrt{2/(\pi y)} F(\beta y), \quad (0 < y < b), \quad F(y) = -V_1 + V_2, \quad G(\beta y) = (2V_2 / h) \sum_{n=1}^{\infty} (-1)^{n-1} \cos(\mu_n y) / \lambda_n^2. \]
\[ E_0 a_2(y) + \sum_{m=1}^{\infty} E_m \sum_{n=1}^{\infty} \frac{J_{v+2m+1+p/2}(\mu_n) J_v(\mu_n y)}{\mu_n^{2+p/2} J_{v+1}^2(\mu_n^2)} = \sqrt{2/(\pi \beta^2)} G(\beta y), \quad (b \leq y < 1), \]  

(18)

where

\[ a_j(y) = \sum_{n=1}^{\infty} \frac{J_{v+1+p/2}(\mu_n) J_v(\mu_n y)}{\mu_n^{2+p/2} J_{v+1}(\mu_n^2)} \quad \text{for} \; j=1,2. \]  

(19)

The reduced model of eqs. (17) and (18) is processed further using specific expressions that relate the infinite series of Bessel functions which appear in eqs. (17) and (18) and the Sonine-Schaafheitlin integral. In particular it can be shown that

\[ \int_0^{\infty} u^{1/2} J_{2m}(u) J_{-1/2}(yu) \, du = \sqrt{2/y} F_1(m+1/2,-m+1/2;1/2; y^2) / \Gamma(2m+1), \]  

(20)

\[ \int_0^{\infty} u^{1/2} J_{2m}(u) J_{-1/2}(yu) \, du = \sqrt{1/2} y^{m-1/2} F_1(m-m;1/2; y^2) / \Gamma(2m+1), \]  

(21)

where \( F_1 \) is the hypergeometric function with single variable. Finally, after further mathematical manipulations the dual trigonometrical series of eqs. (17) and (18) are reduced to the more compact forms

\[ \chi'(\psi) \psi_0 / 2 + \sum_{m=1}^{\infty} \psi_m \cos(m \psi) = F^*(\psi), \quad (0 < \psi < d^*), \]  

(22)

\[ \xi_0 / 2 + \sum_{m=1}^{\infty} \xi_m \cos(m \psi) = G^*(\psi), \quad (d^* < \psi < \pi). \]  

(23)

The new unknowns of the problem are now the coefficients \( \zeta_m \), whilst the parameters \( \psi, \chi(\psi), F^*(\psi), G^*(\psi) \) depend only on the given data of the problem (via expressions omitted for brevity). Dual trigonometrical series of the form of eqs. (22) and (23) have been treated by several authors (see Sneddon [4]).

The coefficients \( \zeta_m \) are

\[ \zeta_0 = \frac{2}{\pi} \left[ \frac{1}{\sqrt{2}} \int_0^{d^*} h_1(t) dt + \frac{\pi}{d^*} \int_0^{d^*} G^*(t) dt \right] \]  

(24)

\[ \zeta_m = \frac{2}{\pi} \left[ \frac{1}{2\sqrt{2}} \int_0^{d^*} h_1(t) \left( P_n \cos(t) + P_{n-1} \cos(t) \right) dt + \frac{\pi}{d^*} \int_0^{d^*} G^*(t) \cos(mt) dt \right], \; m=1,2,3,\ldots \]  

(25)

where \( P_n \) denotes the Legendre polynomial of degree \( n \) and

\[ h_1(t) = \frac{2}{d} \int_0^t \frac{\sin(u/2)}{\cos(u) - \cos(t)} \left[ - \frac{dF^*(u)}{du} + \frac{1}{2} \zeta_0 \frac{d\chi(u)}{du} - \frac{2}{\pi} \sum_{m=1}^{\infty} \left( \frac{\pi}{d^*} \int_0^{d^*} G^*(v) \cos(mv) dv \right) \sin(mu) \right] du. \]  

(26)

4 Some results

Although eqs. (24)-(26) appear in a compact set of expressions, the coefficient \( \zeta_0 \) lies within the definition of the function \( h_1(t) \). The right-hand side of eq. (24) depends on \( \zeta_0 \), so this coefficient can be computed after rearrangement of terms or by an iterative scheme. Once \( \zeta_0 \) is accurately evaluated, expression (26) is wholly determined and can be used in eq. (25) to compute the other coefficients. Also, the truncation of the infinite series expressions has to be handled sensitively to ensure that eq. (16) is satisfied accurately and at least 200 modes were found necessary to achieve an accuracy of 4 significant digits for the coefficients \( \zeta_m \). The rapid decrease in magnitude of \( \zeta_m \), with increasing \( m \), is shown in Fig. 2.

In the following we present some results for wave impacts for which we hold fixed \( h=1m, \beta=0.8m, \) \( V_2=(gh)^{1/2} \approx 3m/s \) and vary \( \alpha \) and \( V_1 \). The solution process indeed converges quite fast, not only for the leading
term $\zeta_0$ but also for the other expansion coefficients $\zeta_m$, $m > 0$. Figs. 3 and 4 show the pressure impulse
$$p = -\rho \phi$$
[see eq. (8)] for unit density $\rho$, along the lower portion of the wall for several $\alpha$ (Fig. 3) and several
values of the velocity $V_1$ (Fig. 4). As expected the pressure impulse is reduced for lower velocities. An
interesting result is that the pressure impulse obtains its maximum value at the bottom. Also, for the same
velocities the pressure impulse is decreased for longer wetted sections (Fig. 3). Further fluid mechanical
consequences of the results will be presented in the talk.

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References
Company, Amsterdam.
Mathematics, 12(3), 379-386.

Figure 2 The expansion coefficients $\zeta_m$, $m > 0$ for $h=1$, $\alpha=0.1$, $\beta=0.8$ and $V_1=V_2=3$ where $\zeta_0 = -1.9744$.

Figure 3 Normalized pressure impulse for equal velocities $V_1=V_2=3$ and variable $\alpha=0.1$, 0.3 and 0.5.

Figure 4 Normalized pressure impulse for $\alpha=0.1$ and variable velocity $V_1=3$, 2 and 1.