An extended multi-modal expansion for propagation of waves in a channel of non-uniform width

NICK R. T. BIGGS

Department of Mathematics and Statistics,
School of Mathematics and Physical Sciences,
University of Reading, P. O. Box 220,
Whiteknights, Reading, RG6 7AX, U.K.

(n.r.t.biggs@reading.ac.uk)

Highlights:

- Taking the appropriate limit of the solution of the Helmholtz equation in a wedge geometry suggests a novel set of cross-channel expansion functions, used to model wave propagation in a channel of non-uniform width, which depend on both the local channel width and the slope of the channel walls.
- Numerical results for an extended multi-modal expansion incorporating these novel expansion set will be presented at the Workshop.

1. Introduction

In 2005, Ehrenmark derived in [1] a new dispersion relation for linear surface gravity waves on finite-depth fluid, extending the classical formula $\omega^2/g = k\tanh(kh)$, in which ω is the prescribed angular wave frequency, h is the constant depth, k is the wavenumber and g is the acceleration due to gravity, to the case in which the fluid bed is sloping linearly, with $h' \equiv \text{constant} = \tan\alpha$, say, for α small. He found that

$$\omega^2/g = k\tanh((\alpha\cot\alpha)kh), \tag{1.1}$$

in which the combination $\alpha\cot\alpha \to 1$ in the limit as $h' \to 0$, so recovering the classical result as the bed flattens out. This result comes from a two-term asymptotic expansion, for small α, of an integral form of the standing wave solution on a plane beach, using the method of steepest descent. In [1] this extended dispersion relation is then used within various forms of the mild-slope equation (MSE) to derive numerical solutions which agree well with numerical solutions of the full linear problem, even for relatively steep bed slopes.

Here we follow a similar procedure, but applied to the propagation of linear surface gravity waves along a uniform-depth channel, bounded by vertical walls at $y = \pm h_\pm(x)$, with x and y denoting (horizontal) Cartesian coordinates, which vary with x. If the channel walls vary linearly with x then the domain is wedge-shaped (analogous to Ehrenmark's plane beach geometry); the solution is easily found explicitly in terms of appropriate polar coordinates, with azimuthal dependence proportional to $\cos[\mu_n(\theta-\theta_1)]$ for $n \in \mathbb{N}_0$, where $\mu_n = n\pi/(\theta_2-\theta_1)$ for particular θ_1, θ_2. The appropriate limit of this solution recovers at leading order the uniform width solution, whose cross-channel dependence is proportional to $\cos[\alpha_n(y+h_-)]$ for $n \in \mathbb{N}_0$, where $\alpha_n = n\pi/w$, in which $w(x) = h_-(x) + h_+(x)$ is the channel width. But if the full unapproximated expression for μ_n is retained, the cross-channel eigenvalue α_n is replaced by

$$\bar{\alpha}_n = \alpha_n \left(w' \Big/ \tan^{-1}\left(\frac{w'}{1-h'_-h'_+}\right)\right).$$

This suggests that a multi-modal approximation in which the solution is expanded in terms of $\cos[\bar{\alpha}_n(y + h_-)]$ may yield accurate results, in the same way as the MSE incorporating (1.1) performs well, and this we investigate here.

2. Statement of problem

We consider the propagation of linear surface gravity waves in a channel of uniform depth h. Cartesian coordinates (x, y, z) are used, z being measured vertically upwards from the undisturbed free-surface. The channel occupies the region $-h_-(x) < y < h_+(x)$, for given continuous functions $h_\pm(x)$, so that we require the solution $\phi(x, y)$ of the boundary-value problem

$$\begin{cases} \phi_{xx} + \phi_{yy} + k^2\phi = 0 & (-h_-(x) < y < h_+(x)) \\ \phi_y \mp h'_\pm \phi_x = 0 & (y = \pm h_\pm(x)) \end{cases} \tag{2.2}$$

together with appropriate radiation conditions. The wavenumber k is the positive root of the dispersion relation $\omega^2 = gk \tanh(kh)$, in which $\omega > 0$ is the prescribed angular wave frequency (harmonic time-dependence proportional to $e^{-i\omega t}$ is implicit throughout) and g is the acceleration due to gravity.

2.1 Multi-modal expansion

In regions of constant width, the solution of (2.2) can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty}(A_n^+ e^{i\gamma_n x} + A_n^- e^{-i\gamma_n x})\phi^{(n)}(h_\pm, y), \quad \phi^{(n)}(h_\pm, y) = \cos[\alpha_n(y + h_-)], \tag{2.3}$$

where $A_n^\pm$ are constants, $w = h_+ + h_-$ is the channel width, $\alpha_n = n\pi/w$, and

$$\gamma_n = \begin{cases} \sqrt{k^2 - \alpha_n^2} & \text{if } k \geq \alpha_n, \\ i\sqrt{\alpha_n^2 - k^2} & \text{if } k < \alpha_n, \end{cases}$$

but in regions where the channel width varies with x a multi-mode expansion of some sort is commonly used to approximate the solution (see [2] for a sophisticated example). The simplest multi-mode expansion assumes that the local modal structure can be approximated by that of a uniform width channel of the same (local) width, so that we write

$$\phi \approx \bar{\phi} = \sum_{n=0}^{M} v_n(x)\phi^{(n)}(h_\pm(x), y) \tag{2.4}$$

for some prescribed $M \in \mathbb{N}_0$, and where now $h_\pm = h_\pm(x)$. Because this approximation cannot hope to exactly satisfy (2.2) we instead require that

$$\int_{-h_-}^{h_+} \left(\bar{\phi}_{xx} + \bar{\phi}_{yy} + k^2\bar{\phi}\right) \phi^{(m)}(h_\pm(x), y)\, \mathrm{d}y = 0, \qquad (m = 0, 1, \ldots, M).$$

This yields a system of differential equations of the form

$$A(x)\mathbf{v}''(x) + 2B(x)\mathbf{v}'(x) + C(x)\mathbf{v}(x) = \mathbf{0},$$

in which $\mathbf{v} = (v_0, \ldots, v_M)^T$, and A, B and C are known matrix-valued functions whose entries are integrals of combinations of the $\phi^{(m)}$ and their derivatives.

2.1.1 Solutions in the wedge geometry, and its limiting form

As an extension of the standard multi-mode expansion, we seek to approximate the local modal structure by that of the solution for a linearly widening or narrowing channel, i.e. for a *wedge*.

Consider the channel close to the point $x = x_0$. For x near x_0, we have $h_\pm(x) \approx h_\pm(x_0) + h'_\pm(x_0)(x - x_0)$, and provided $w'(x_0) \neq 0$ the two straight lines $y = \pm[h_\pm(x_0) + h'_\pm(x_0)(x - x_0)]$ meet at $(x, y) = (\bar{x}, \bar{y})$ where

$$\bar{x} = x_0 - \frac{w(x_0)}{w'(x_0)}, \quad \bar{y} = \frac{h_+(x_0)h'_-(x_0) - h'_+(x_0)h_-(x_0)}{w'(x_0)}.$$

If the channel is locally widening (narrowing) with increasing x then $\bar{x} < x_0$ ($\bar{x} > x_0$). These straight lines form the boundaries of our wedge-shaped domain, and $(\bar{x}, \bar{y})$ is its apex. In terms of polar coordinates (r, θ) defined by $r^2 = (x - \bar{x})^2 + (y - \bar{y})^2$, $\tan\theta = (y - \bar{y})/(x - \bar{x})$, the solution of the Helmholtz equation in the domain $\{(r, \theta) : r > 0,\ \theta_1 < \theta < \theta_2\}$ subject to homogeneous Neumann conditions on $\theta = \theta_1, \theta_2$ for $r > 0$ can be written as

$$\phi = \sum_{n=0}^{\infty} [A_n J_{\mu_n}(kr) + Y_{\mu_n}(kr)] \cos[\mu_n(\theta - \theta_1)], \quad \mu_n = n\pi/(\theta_2 - \theta_1), \tag{2.5}$$

where J_{μ_n} and Y_{μ_n} denote Bessel functions of order μ_n and first and second kind, respectively. Here the boundaries of the wedge are

$$\theta_1 = \tan^{-1}\left(\frac{-h_-(x_0) - \bar{y}}{x_0 - \bar{x}}\right), \quad \theta_2 = \tan^{-1}\left(\frac{h_+(x_0) - \bar{y}}{x_0 - \bar{x}}\right),$$

and we've assumed that the channel is locally widening, so that $\bar{x} < x_0$; a similar expression results if the channel is locally narrowing.

In the limit as the channel walls straighten out, we recover from (2.5) the uniform width solution (2.3). To see this, write $h'_\pm(x_0) = \epsilon\bar{h}'_\pm(x_0)$ and $w'(x_0) = \epsilon\bar{w}'(x_0)$, in which $0 < \epsilon \ll 1$ and $\bar{h}'_\pm(x_0), \bar{w}'(x_0) = O(1)$, and consider the limit $\epsilon \to 0$. Then

$$r = \left[(x - \bar{x})^2 + (y - \bar{y})^2\right]^{1/2} = \frac{w}{\epsilon\bar{w}'} + (x - x_0) + O(\epsilon)$$

and $\mu_n = n\pi/\epsilon\bar{w}' + O(\epsilon)$, in which $w, \bar{w}'$ etc. are all evaluated at $x = x_0$, so that

$$J_{\mu_n}(kr) \sim J_{n\pi/\epsilon\bar{w}'}\left(\frac{n\pi}{\epsilon\bar{w}'} \times \frac{k}{n\pi}[w + \epsilon\bar{w}'(x - x_0)]\right),$$

and similarly for $Y_{\mu_n}(kr)$. If $kw/n\pi < 1$ then use of Debye's asymptotics for Bessel functions of large argument and order (e.g. [3, Eq. (9.3.2)]) shows that

$$J_{\mu_n}(kr) \sim \text{constant} \times \exp(x\sqrt{\alpha_n^2 - k^2}),$$

which agrees with the form of the modes (2.3) which grow as x increases. A similar calculation for $Y_{\mu_n}(kr)$ recovers the modes which decay with increasing x. (If $n = 0$, the standard expansions of J_0 and Y_0 for large argument yields the analogous results.) If instead $kw/n\pi > 1$ then use of [3, Eq. (9.3.3)] gives

$$J_{\mu_n}(kr) \sim \text{constant} \times \cos[(x - x_0)\sqrt{k^2 - \alpha_n^2}]$$

and

$$Y_{\mu_n}(kr) \sim \text{constant} \times \sin[(x - x_0)\sqrt{k^2 - \alpha_n^2}]$$

as $\epsilon \to 0$, which can be combined to give the form of the propagating modes from (2.3). (For simplicity we suppose here that k doesn't coincide with a cut-off frequency α_n.)

More interesting for our purposes is the behaviour of the azimuthal terms in (2.5) in the limit $\epsilon \to 0$. With a bit of rearranging, we have

$$\begin{aligned} \mu_n &= n\pi \Big/ \tan^{-1}\left(\frac{\epsilon \bar{w}'}{1-\epsilon^2 \bar{h}'_- \bar{h}'_+}\right) && (2.6) \\ &= \frac{n\pi}{\epsilon \bar{w}'} + O(\epsilon), && (2.7) \end{aligned}$$

so that, when combining (2.7) with the behaviour

$$\theta - \theta_1 = \frac{\epsilon \bar{w}'}{w}(y + h_-) + O(\epsilon^2)$$

we see that

$$\cos[\mu_n(\theta - \theta_1)] \sim \cos[n\pi(y + h_-)/w]$$

as $\epsilon \to 0$, which agrees with the form of $\phi^{(n)}$ in (2.3). However, retaining the full expression (2.6) yields the approximate cross-channel structure

$$\cos[\mu_n(\theta - \theta_1)] \approx \cos[\bar{\alpha}_n(y + h_-)], \tag{2.8}$$

where

$$\begin{aligned} \bar{\alpha}_n &= \alpha_n \left(\epsilon \bar{w}' \Big/ \tan^{-1}\left(\frac{\epsilon \bar{w}'}{1-\epsilon^2 \bar{h}'_- \bar{h}'_+}\right)\right) \\ &\equiv \alpha_n \left(w' \Big/ \tan^{-1}\left(\frac{w'}{1-h'_- h'_+}\right)\right). && (2.9) \end{aligned}$$

In (2.9), α_n is recovered if the width w is unchanging with x.

2.1.2 An extended multi-modal expansion

Given (2.9), we propose an extended multi-modal expansion

$$\phi \approx \bar{\phi} = \sum_{n=0}^{M} v_n(x) \bar{\phi}^{(n)}(h_\pm(x), h'_\pm(x), y), \quad \bar{\phi}^{(n)}(h_\pm, h'_\pm, y) = \cos[\bar{\alpha}_n(y + h_-)]. \tag{2.10}$$

Numerical results using this expansion will presented at the Workshop.

References

[1] Ehrenmark, U.T. 2005 An alternative dispersion relation for water waves over an inclined bed. *J. Fluid Mech.* **543**, 249-266.

[2] Felix, S. & Pagneux V. 2001 Sound propagation in rigid bends: a multimodal approach. *J. Acoust. Soc. Am.* **110** (3 Pt 1), 1329-1337.

[3] Abramowitz, M. & Stegun, I. A. 1965 *Handbook of Mathematical Functions*. Dover Publications, New York.