A New Method for the Integration of the Transient Green Function over a Panel

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Highlights: A new method is described to evaluate the transient Green function over a quadrilateral panel. This method is more accurate than numerical integration schemes. The method greatly improves the computational efficiency of time domain panel methods for seakeeping simulations.

1. Introduction

In time domain panel methods the transient Green function can be used to account for free surface effects. The Green function and its derivatives need to be integrated both in space (over the panel area) and in time. This process requires a large amount of computer time, especially in the body non-linear method where at each time step the complete convolution integral has to be computed.

Furthermore, it is difficult to achieve sufficient accuracy by means of numerical quadrature due to the highly oscillating nature if the Green function. This is especially true for panels located close to the free surface and for waterline elements.

This paper describes a more analytical integration method that improves the accuracy and also the computational efficiency.

2. Integral expressions

It is assumed that the panel method uses quadrilateral surface panels with a constant strength source and/or doublet strength. The free surface Green function integrated over the panel area S is defined by G^P:

$$G^P = \iint_S G_s^f \, dS = 2\int_0^t d\tau \iint_S d\xi d\eta \int_0^\infty \sqrt{k}\sin(\sqrt{k}(t-\tau))\; e^{k(z+\zeta)} J_0(kr)dk \tag{1}$$

here t is time, τ is past time, k is the wave number, J_0 is the Bessel function of order 0 and S is the panel area. All physical quantities used are made non-dimensional on basis of a reference length and the gravitational constant. Furthermore:

$$\begin{aligned} &x, y, z \text{ : field point coordinates} \\ &\xi, \eta, \zeta \text{ : singularity (panel) coordinates} \\ &r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \text{ : horizontal distance between field and singularity points} \end{aligned} \tag{2}$$

A local panel axis system (ξ_l ,η_l ,ζ_l) with its origin in the panel collocation point is defined. The integration uses local cylindrical coordinates (r_l, θ_l) at each corner point p of the quadrilateral panel. By defining:

$$\begin{aligned} z_r &= (z+\zeta_c) \\ \alpha_2 &= (t_{13}\cos\theta_l + t_{23}\sin\theta_l) \end{aligned} \tag{3}$$

where t_{13} and t_{23} are elements of the coordinate transformation matrix from the local panel reference frame to the global reference frame, and changing the order of the surface and wave number integration, the integral for a single corner point is written as:

$$G^P = 2\int_0^t d\tau \int_0^\infty dk\sqrt{k}\sin\;(\sqrt{k}(t-\tau))\; e^{kz_r} \int_{\theta_{l1}}^{\theta_{l2}} \int_0^{r_l} e^{k\alpha_2 r_l} r_l J_0\;\; kr\;\; dr_l d\theta_l \tag{5}$$

We can express $J_0\,(kr)$ as a function of r_l and r_p by using the addition theorem for Bessel functions:

$$J_0(kr) = J_0(kr_p)\cdot J_0(kr_{l2}) + 2\sum_{\nu=1}^{\infty} J_\nu(kr_p)\cdot J_\nu(kr_{l2})\cdot\cos(\nu\theta) \tag{6}$$

where r_{l2} is the horizontal component of r_l and θ is the angle between r_{l2} and r_c. Inserting this expression in Eq. (5) and summing over the four corner points yields the following expression:

$$G^P = 2\sum_{i=1}^{4}\int_0^t d\tau \int_0^\infty \sqrt{k}\sin(\sqrt{k}(t-\tau))e^{kz_r}dk \int_{\theta_{li1}}^{\theta_{li2}} d\theta_l \int_0^\infty e^{\alpha_2 kr_l} r_l \sum_{\nu=1}^{\infty} J_\nu(kr_{pi})J_\nu(k\alpha_3 r_l)\cos(\nu\theta)dr_l \tag{7}$$

The surface integral Eq. (7) is the sum of four integrals, one over each corner point, whereby the radius r_l is taken from zero to infinity. The first corner point (1) is the point with the maximum (z+ζ) value such that the constant α_2 is negative and the integral exists. For the integrals over corner points 2 and 3 a minus sign must be added. The integral over the last corner point (4) is positive again to compensate for the negative parts in the integrals over points 2 and 3. The radial integration limits are θ_{li1} and θ_{li2}, for corner point i.

After a lengthy process involving recurrence relations and partial integration G^P can be expressed by means of a double and a single integral for each corner point:

$$G^P_\lambda = 4c_1\int_0^\infty \sin \lambda t \; e^{\delta\lambda^2} \frac{d\lambda}{\lambda^2}\int_0^{r_p\lambda^2} e^{-cs} J_0(s)ds + 4c_2\int_0^\infty \sin \lambda t \quad e^{\delta\lambda^2} - e^{\alpha\lambda^2} \; \frac{d\lambda}{\lambda^2}$$

where α is a desingularisation constant, $\delta = z_r + cr_p$, c_1, c_2 and c are panel geometry constants (8)

and the substitution $k = \lambda^2$ has been applied.

3. Initial value problem

By means of partial integration the integrals in Eq. (8) can be shown to satisfy the initial value problem:

$$\frac{d^2G^P_\lambda}{d\tilde{t}^2} - \frac{t}{2\delta}\frac{dG^P_\lambda}{dt} + \frac{G^P_\lambda}{2\delta} = F(t), \text{ with}$$

$$F(t) = \frac{4}{\delta}\int_0^\infty \sin \; t\lambda \;\; e^{z_r\lambda^2}\left(r_p c_1 J_0(r_c\lambda^2) - (\alpha-\delta)c_2 e^{\;\alpha - z_r \;\lambda^2}\right)d\lambda \qquad (9)$$

$$= \frac{c_1 r_p}{\delta}G^f_s - \frac{4}{\delta}c_2(\alpha-\delta)\int_0^\infty \sin \; t\lambda \;\; e^{\alpha\lambda^2} d\lambda$$

The material second time derivative can be expressed as:

$$\frac{DG^P_{\lambda\, tt}}{D\tilde{t}} = \frac{1}{2\delta}\left(\tilde{t} - \delta_t - \frac{\tilde{t}^2\delta_t}{2\delta}\right)G^P_\lambda + 4c_1B_1 + 4c_2B_2$$

$$B_1 = \frac{c_t}{8(1+c^2)}G^f_s - \left(\frac{r_p t\delta_t}{8\delta^2} - \frac{r_p}{4\delta} - \frac{tc_t}{8(1+c^2)}\right)G^f_c - \left(\frac{r_p\delta_t}{4\delta} - \frac{r_{pt}}{4} + \frac{c_t(z_r - r_p c)}{4(1+c^2)}\right)G^f_{ct}$$

$$B_2 = -\left(\frac{(\alpha-\delta)}{\delta}\left(1 - \frac{\tilde{t}\delta_t}{2\delta}\right)\right)\frac{DF_\alpha}{Dt} + \delta_t\left(1 + \frac{\alpha-\delta}{\delta}\right)\frac{D^2F_\alpha}{D\tilde{t}^2} \qquad (10)$$

$$F_\alpha = \int_0^\infty \sin(\tilde{t}\lambda)\; e^{\alpha\lambda^2} d\lambda$$

$$\frac{D^2F_\alpha}{D\tilde{t}^2} = \frac{1}{\alpha}\left(F_\alpha + \frac{\tilde{t}}{2}\frac{DF_\alpha}{D\tilde{t}}\right)$$

$$G^P_{\lambda\, tt}(0) = 0$$

the subscript t denotes the derivative with respect to time $\tilde{t}$.

Three basic Green functions present in Eq. (9) and (10) are defined by:

$$G^f_s = 4\int_0^\infty \sin(\tilde{t}k)\; e^{z_c k^2} J_0(r_p k^2)\; dk, \; G^f_c = \frac{dG^f_s}{d\tilde{t}} = 4\int_0^\infty k\cos(\tilde{t}k)\; e^{z_c k^2} J_0(r_p k^2)\; dk \text{ and}$$

$$G^f_{ct} = \frac{d^2G^f_s}{d\tilde{t}^2} = -4\int_0^\infty k^2 \sin(\tilde{t}k)\; e^{z_c k^2} J_0(r_p k^2)\; dk \qquad (11)$$

These three Green functions can be expressed in a separate initial value problem. For the zero forward speed case Clement [1] provides the initial value problem for G^f_s. For forward speed conditions, the material derivatives can be obtained as well and the initial value problems for G_s^f, G_c^f and G_{ct}^f can be solved simultaneously with the initial value problem of the integral of G_s^f over the panel area, Eq. (10). Time integrals and derivatives can be obtained at the same time. This approach is efficient and accurate for not too small |δ| values. For small |δ| values (typically |δ|<0.05) the efficiency and accuracy deteriorate and the series expansions described in the next section must be used.

4. Series expansions - Small β values

For series expansions the integral expression for $G_{\lambda\, tt}^P$ in Eq. (9) is the starting point, expressed in terms of the non-dimensional (μ, β, δ, α) parameters.

$$\begin{aligned}
\frac{d^2 G_\lambda^P}{d\beta^2} &= e^{\beta^2/4\tilde{\delta}} \int_0^\beta e^{-s^2/4\tilde{\delta}} \frac{dF(s)}{ds} ds \text{ , with} \\
\frac{dF(s)}{ds} &= \frac{4}{\tilde{\delta}} \int_0^\infty \lambda \cos(s\lambda)\, e^{\mu\lambda^2} \left(c_1 \sigma J_0(\sigma\lambda^2) - c_2 (\tilde{\alpha} - \tilde{\delta})\, e^{(\tilde{\alpha}-\mu)\lambda^2} \right) d\lambda \\
&= \frac{1}{\tilde{\delta}} \left[c_1 \sigma G_c^f(\mu, s)\, ds - 4 c_2 (\tilde{\alpha} - \tilde{\delta}) \int_0^\infty \lambda \cos(s\lambda)\, e^{\tilde{\alpha}\lambda^2} d\lambda \right], \text{ with} \\
\tilde{\alpha} &= \frac{\alpha}{R_0}, \tilde{\delta} = \frac{\tilde{\delta}}{R_0}, \sigma = \sqrt{1-\mu^2}, \mu = \frac{z_r}{R_0}, \beta = \frac{\tilde{t}}{\sqrt{R_0}}, R_0 = \sqrt{x_r^2 + y_r^2 + z_r^2}
\end{aligned} \tag{12}$$

The cosine term in the infinite integral in Eq. (12) can be expanded in an infinite series yielding:

$$\int_0^\infty \lambda \cos(s\lambda)\, e^{\tilde{\alpha}\lambda^2} d\lambda = \sum_{n=0}^\infty (-1)^n \frac{s^{2n} n!}{(2n)!} \int_0^\infty \lambda^n e^{\alpha\lambda} d\lambda = \sum_{n=0}^\infty (-1)^n \frac{s^{2n} n!}{(2n)!} \frac{1}{(-\alpha)^{n+1}} \tag{13}$$

Using expansions for $G_c^f(\mu,\beta)$ developed by Newman [2] and combing the two terms in to a single series expansion yields:

$$G_{\lambda\,\beta\beta}^P = \frac{2}{\delta} e^{\beta^2/4\tilde{\delta}} \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n)!} \left[c_1 \sigma P_n(\mu) - c_2 \frac{\alpha - \delta}{(-\alpha)^{n+1}} \right] \int_0^\beta e^{-s^2/4\tilde{\delta}} s^{2n} ds \tag{14}$$

Recurrence relations can be used to evaluate this infinite series. Integrals and derivatives of Eq. (12) can be evaluated in a analogous fashion.

5. Series Expansions - Large β values

The starting point is the same integral as used in Eq. (12), but now written as an indefinite integral:

$$\frac{d^2 G_\lambda^P}{d\beta^2} = e^{\beta^2/4\tilde{\delta}} \int_\beta e^{-s^2/4\tilde{\delta}} \frac{dF(s)}{ds} ds \tag{15}$$

For 'large' β values (β>9) the asymptotic expansion for $G_c^f(\mu,\beta)$ developed by Newman [2] is used.

$$\begin{aligned}
G_c^f(\mu,s) &= G_{c1}^f(\mu,s) + G_{c2}^f(\mu,s) \\
&= -4 \sum_{n=0}^\infty \frac{(2n+1)!}{n!} P_n(\mu)\, s^{-(2n+2)} \\
&\quad + 4 \sqrt{\frac{\sigma + i\mu}{2\sigma}} e^{-s^2/4\delta_l} \sum_{n=0}^\infty c_n \left(\frac{i}{\sigma}\right)^n \sum_{m=0}^\infty d_{nm} \left(\frac{is}{2}\right)^{-2n-2m} \delta_l^{2n+m}
\end{aligned} \tag{16}$$

where $\delta_l = \mu + i\sigma$, $c_n = \dfrac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\pi 2^n n!}$ and $d_{nm} = \dfrac{(2n+2m-1)!}{2^{2m} m!(2n-1)!}$

For the second term in Equation (12) an asymptotic expansion can be derived through the use of Watson's Lemma [3]:

$$\int_0^\infty \cos(s\lambda)\,\lambda e^{\alpha\lambda^2} d\lambda = -\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!}{n!} \alpha^n s^{-2(n+1)} \tag{17}$$

Substitution of these expansions for $G^f_{c1}(\mu,s)$ and $G^f_{c2}(\mu,s)$ and Eq. (17) in Eq. (15) yields:

$$\frac{d^2 G^P_\lambda}{d\beta^2} = -\frac{4}{\tilde{\delta}} \sum_{n=0}^{\infty} \frac{(2n+1)!}{n!}\left[c_1\sigma P_n(\mu) - c_2(\tilde{\alpha}-\tilde{\delta})(-1)^n \alpha^n\right] e^{\beta^2/4\tilde{\delta}} \int_\beta e^{-s^2/4\tilde{\delta}} s^{-(2n+2)} ds +$$

$$2e^{\beta^2/4\tilde{\delta}} \gamma \sum_{n=0}^{\infty} c_n \left(\frac{i}{\sigma}\right)^n \sum_{m=0}^{\infty} (-4)^{n+m} d_{nm} \tilde{\delta}_l^{2n+m} \int_\beta^\infty e^{-s^2/4\delta_s} s^{-2(n+m)} ds + 4c_2 \frac{\sqrt{\pi}}{\sqrt{\tilde{\delta}}} e^{\beta^2/4\tilde{\delta}}, \text{ with} \tag{18}$$

$\gamma = \dfrac{c_1\sqrt{2\sigma}\sqrt{\sigma \pm i\mu}}{\tilde{\delta}}$, $\delta_s = \dfrac{\tilde{\delta}\tilde{\delta}_l}{\tilde{\delta}+\tilde{\delta}_l}$ and $\tilde{\delta}_l = \mu \pm i\sigma$

The terms in these summations can again be obtained efficiently via recursion.

6. Application

For a typical "difficult" case a comparison is given between the new method (G-anal) and the existing Gaussian quadrature method (G-num). The conditions are:

- Panel dimensions 2x2 m, centroid position (ξ_0,η_0,ζ_0) = (0.0, 0.0, -0.01), panel normal vector n=(0.10, 0.0, 0.995)
- Initial position field point (x_0,y_0,z_0) = (-5.0, 2.0, -0.01)
- Field point travelling with a speed U of 2.5 m/s in positive x-direction ($x=x_0+Ut$)
- Gaussian quadrature: 16 sub panels, on each sub panel a 9-point integration rule is applied.

Figures 1 and 2 show that results for G (Eq. 1) are fairly similar, except around t = 2 sec, i.e. when the field point is close to the panel. The more analytical method shows a much smoother behaviour than the numerical method.

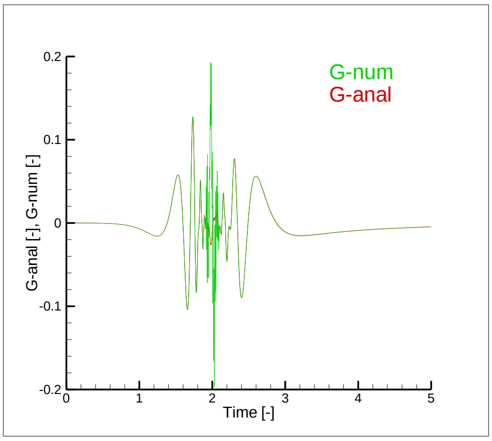

Figure 1 Comparison *G*-functions

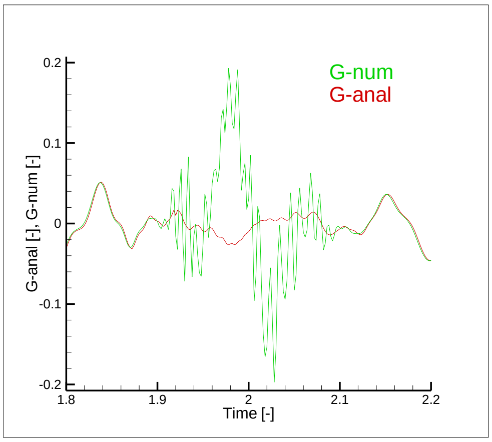

Figure 2 Comparison *G*-functions - detail

7. References

[1] Clement, A.H., 'An ordinary differential for the Green function of time-domain free-surface hydrodynamics', Journal of Engineering Mathematics 33: 201-217, 1998.

[2] Newman J.N., 'The approximation of free surface Green functions', Wave Asymptotics, pp. 107-135, Cambridge University Press, 1992.

[3] Watson, G. N., "The harmonic functions associated with the parabolic cylinder", Proceedings of the London Mathematical Society 2 (17), pp 116–148, 1918.