Time-Domain Hydroacoustic Green Function for Surface Pressure Disturbance

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Highlights:

- An analytical solution for the transient potential flow field generated by a localised surface pressure disturbance in a weakly compressible ocean is provided.
- Asymptotic expansions are obtained at large distance from the source.
- Applications are envisaged in the field of early storm surge detection.

1. Introduction

At the last Workshop, Dai & Chen (2013) presented a study on the potential flow generated by an impulsive point source with surface tension. Stimulated by their approach, and motivated by the fact that water compressibility is usually neglected in the traditional potential flow theory, in this paper we investigate the effect of introducing slight density variations in the initial-value problem of water waves generated by a sudden impulsive pressure disturbance, acting on the free-surface of an ocean of constant depth. When water compressibility is considered, impulsive actions generate not only transient travelling surface waves, but also fast-moving pressure disturbances in the form of hydroacoustic waves. Because of their property of travelling much faster than the surface waves, such hydroacoustic signals could be exploited for the early detection of incoming free-surface disturbances, like for example storm surges. A similar technique has been suggested by Stiassnie (2010) for the early detection of tsunamis generated by an impulsive bottom displacement. However, to our knowledge, nothing as such has been envisaged before for the early detection of waves generated by impulsive pressure disturbances acting on the free surface.

2. Mathematical Model

Consider an ocean of constant depth h in two dimensions. Let x be the horizontal axis and z the vertical axis, positive upwards. Let $z = 0$ denote the unperturbed free surface of the ocean and $z = -h$ the seafloor; t is time. Assume an inviscid and slightly compressible fluid of density $\rho(x, z, t) = \rho_0 + \rho'(x, z, t)$, with $\rho' \ll \rho_0$. Following a strategy devised by Mei *et al.* (2005), we shall build the transient potential response $\Phi(x, z, t)$ to the pressure disturbance $P_a(x, t) = P_0\delta(x)\delta(t)$ acting on the free surface, by starting from the complex outgoing steady-state solution $\phi(x, z)$:

$$\Phi(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(x, z)e^{-i\omega t}\, dt. \tag{1}$$

Following Yamamoto (1982) and Ardhuin & Herbers (2013), the linearized governing equations for ϕ in a slightly compressible fluid are as follows:

$$\begin{cases} c^2\nabla^2\phi + \omega^2\phi = 0 \\ g\,\phi_z - \omega^2\phi = \frac{i\omega P_0}{\rho_0}\,\delta(x)\,, \quad z = 0 \\ \phi_z = 0, \quad z = -h \end{cases} \quad , \tag{2}$$

where ϕ must be outgoing as $|x| \to \infty$. In (2) g is the acceleration due to gravity, c is the speed of sound in water, ρ_0 the (assumed constant) ambient density of the unperturbed fluid. Subscripts denote differentiation with respect to the relevant variable. Let us now apply the Fourier transform along x

$$\begin{cases} \bar{\phi}(z;\alpha) = \int_{-\infty}^{\infty} \phi(x,z)\, e^{-i\alpha x}\, dx \\ \phi(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}(x;\alpha)\, e^{i\alpha x}\, d\alpha \end{cases} \tag{3}$$

to the system (2), solve the straightforward ODE boundary-value problem obtained and then transform back into the original variables via the second of (3). These steps yield

$$\phi(x,z) = \frac{i\omega P_0}{\rho_0 g \pi} \int_0^{+\infty} \frac{\cosh \beta(z+h)}{\beta \sinh \beta(z+h) - \frac{\omega^2}{g} \cosh \beta h} \cos \alpha x \;\, d\alpha \;, \tag{4}$$

where $\beta(\alpha,\omega) = \sqrt{\alpha^2 - \omega^2/c^2}$, with $\Re\{\beta\} > 0$. Expression (4) features an improper integral, since the integrand is singular at the poles $\alpha = \pm\alpha_0$, $\alpha = \pm\alpha_n$ and $\alpha = \pm i\tilde{\alpha}_n$ which satisfy the dispersion relationships

$$\beta = \beta_0: \quad \omega^2 = g\beta_0 \tanh \beta_0 h \;; \quad \alpha_0 = \sqrt{\beta_0^2 + \omega^2/c^2}, \quad \forall\omega \tag{5}$$

$$\beta = i\beta_n: \quad \omega^2 = -g\beta_n \tan \beta_n h \;; \quad \alpha_n = \sqrt{\omega^2/c^2 - \beta_n^2}, \quad \beta_n^2 < \omega^2/c^2 \tag{6}$$

$$\beta = i\beta_n: \quad \omega^2 = -g\beta_n \tan \beta_n h \;; \quad \tilde{\alpha}_n = \sqrt{\beta_n^2 - \omega^2/c^2}, \quad \beta_n^2 > \omega^2/c^2 \tag{7}$$

respectively. First consider the case $x > 0$ and $\omega > 0$. Integration of equation (4) is performed along the path of figure 1 in the complex α domain. Note the particular indentation of the contour of figure 1 above the negative real poles and below the positive ones, which guarantees the solution to be outgoing at large distance $x \to \infty$ (see Mei, 1997).

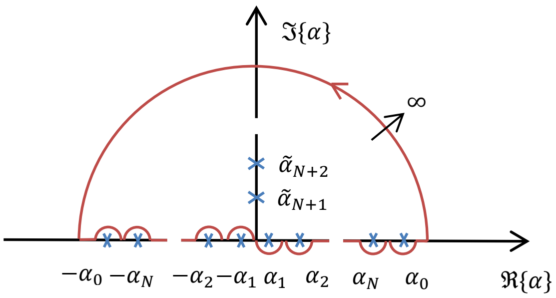

Figure 1. Contour integration in the complex α plane for $\omega > 0$ and $x > 0$. The crosses denote the poles of the integrand in (4).

Note that the integration path we follow in figure 1 is different from that used by Yamamoto (1982) and later adopted by Stiassnie (2010) to obtain the steady-state hydroacoustic waves generated by a moving bottom. Application of the Green integral theorem and the Jordan lemma to (4) yields

$$\phi(x,z) = -\frac{2\omega P_0}{\rho_0 g}\left(\frac{\beta_0 \cosh \beta_0 h \cosh \beta_0(z+h)}{\alpha_0(\sinh 2\beta_0 h + 2\beta_0 h)}\; e^{i\alpha_0 x} + \sum_{n=1}^{N} \frac{\beta_n \cos \beta_n h \cos \beta_n(z+h)}{\alpha_n(\sin 2\beta_n h + 2\beta_n h)}\; e^{i\alpha_n x} - i\sum_{n=N+1}^{\infty} \frac{\beta_n \cos \beta_n h \cos \beta_n(z+h)}{\tilde{\alpha}_n(\sin 2\beta_n h + 2\beta_n h)}\; e^{-\tilde{\alpha}_n x}\right) \tag{8}$$

where N is the last integer such that (6) is valid. A similar procedure is used in the remaining cases ($x > 0$ and $\omega < 0$; $x < 0$ and $\omega > 0$; $x < 0$ and $\omega < 0$) but is not reported here for the sake of brevity. In each case, particular care must be taken in choosing the only integration path which allows for outgoing solutions in the far field. Incidentally, in the limit $c \to \infty$ expression (8) strongly simplifies to

$$\phi_{inc}(x,z) = -\frac{2\omega P_0}{\rho_0 g} \frac{\cosh kh \cosh k(z+h)}{\sinh 2kh + 2kh}\; e^{ikx} \quad , \tag{9}$$

where the traditional dispersion relation $\omega^2 = gk \tanh kh$ is fully recovered. Expression (9) represents the incompressible limit of (8) and is well known in the literature (see e.g. Stoker, 1957; Debnath, 1969).

After having obtained the steady-state solution $\phi(x,z)$ (8) for all possible combinations of sign(x) and sign(ω), usage of (1) yields finally the sought expression for the Green function:

$$
\begin{aligned}
\Phi(x,z,t) = -\frac{2P_0}{\pi\rho_0 g}\Bigg(&\int_0^{+\infty} \frac{\beta_0\cosh\beta_0 h\cosh\beta_0(z+h)}{\alpha_0(\sinh 2\beta_0 h + 2\beta_0 h)}\ \omega\cos(\alpha_0|x| - \omega t)\ d\omega \\
&+ \sum_{n=1}^{+\infty}\int_{\omega_n}^{+\infty} \frac{\beta_n\cos\beta_n h\cos\beta_n(z+h)}{\alpha_n(\sin 2\beta_n h + 2\beta_n h)}\,\omega\cos(\alpha_n|x| - \omega t)\ d\omega \\
&- \sum_{n=1}^{\infty}\int_0^{\omega_n} \frac{\beta_n\cos\beta_n h\cos\beta_n(z+h)}{\tilde{\alpha}_n(\sin 2\beta_n h + 2\beta_n h)}\ e^{-\tilde{\alpha}_n|x|}\,\omega\sin\omega t\ \ d\omega\Bigg),
\end{aligned}
\tag{9}
$$

where $\omega_n = c\beta_n$. The first term of (9) represents surface gravity waves propagating with wavenumber α_0 along the x axis. The second term represents the propagating hydroacoustic modes, while the last term represents the evanescent hydroacoustic modes, which are important only in the vicinity of $x = 0$ and decay exponentially from the origin.

3. Approximated expressions

Several approximated expressions can be worked out which simplify the complex numerical calculations required to evaluate equation (9). For the sake of brevity, here we shall only derive an approximated formula for the propagating hydroacoustic modes at large distance from the origin. Other useful approximated formulae will be presented at the Workshop. Following Chamberlain and Porter (1999), let us expand

$$
\beta_n h \sim (2n-1)\frac{\pi}{2}\left(1 + \frac{g}{h\omega^2}\right),\ \ n = 1, \ldots, \infty, \tag{10}
$$

and assume

$$
g/(h\omega^2) \ll 1. \tag{11}
$$

Application of the method of stationary phase (see for example Mei *et al.*, 2005) to the second term of (9), together with the approximation dictated by (10), yields the stationary points

$$
\hat{\omega}_n = \frac{\lambda_n\, c/h}{\sqrt{1-\left(\frac{x}{ct}\right)^2}},\quad n = 1, \ldots, \infty, \tag{12}
$$

where $\lambda_n = (2n-1)\pi/2$. Note that the stationary points (12) are the same obtained by Stiassnie (2010) for the hydroacoustic waves generated by a bottom displacement and exist only if $x/(ct) < 1$. With (12), the stationary phase approximation of the propagating hydroacoustic wave in (9) be shown to be

$$
\begin{aligned}
\Phi_c(x,z,t) &\sim \sum_{n=1}^{+\infty}\Phi_{cn}(x,z,t) \\
&= \sum_{n=1}^{+\infty}(-1)^{n+1}\ \frac{P_0}{\rho_0}\sqrt{\frac{2}{\pi ch}}\ t^{-1/2}\ \left[1-\left(\frac{x}{ct}\right)^2\right]^{1/4}\ \lambda_n^{-1/2}\cos\hat{\beta}_n(z+h)\cos\left[\frac{\lambda_n c}{h}t\sqrt{1-\left(\frac{x}{ct}\right)^2}+\frac{\pi}{4}\right],
\end{aligned}
\tag{13}
$$

where $\hat{\beta}_n \sim \frac{\lambda_n}{h}\left(1 + \frac{g}{h\hat{\omega}_n^2}\right)$, $n = 1, \ldots, \infty$. Recall that (13) is valid under the initial assumption (11), which for each of the stationary points (12) now writes $g/(h\hat{\omega}_n^2) \ll 1$. After substitution of (12) in the latter expression, we find that the initial approximation (11) is valid if $gh/c^2 \ll 1$. Since $g \sim 9.81$ m/s^2 and $c \sim 1500$ m/s, the inequality (11) used to obtain the stationary phase approximation (13) for the propagating hydroacoustic modes surely holds for a wide range of water depths.

Let us now have a look at the physical picture given by (13). For an observer moving along the x axis at a constant speed x/t, expression (13) describes transient propagating waves which decay as $O(t^{-1/2})$. Note also that as $x \to ct$, $\Phi_c \to 0$, so that the hydroacoustic potential decays near the wave front. Figure 2 shows a snapshot of the first 2 hydroacoustic normal modes $\Phi_{cn}(x,z,t)$ at large distance from the source, evaluated at the bottom of the ocean. Note that shorter

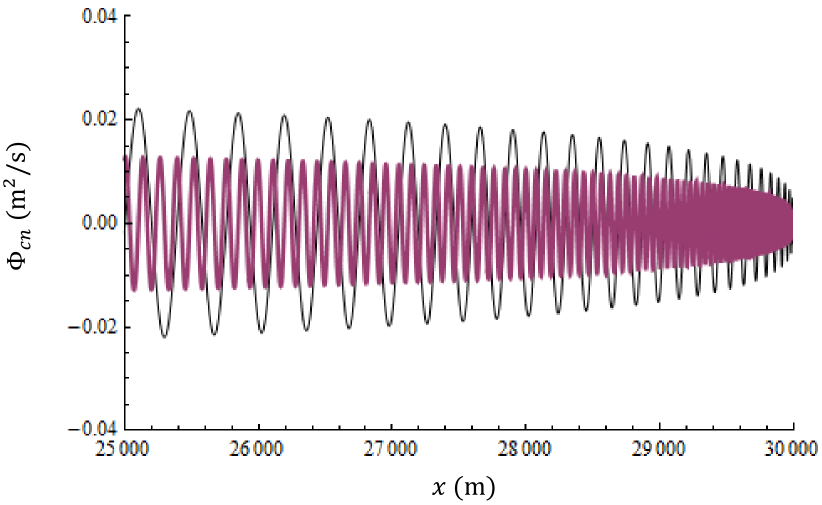

Figure 2. First two hydroacoustic modes at large distance from the source. Thin line: $n = 1$, thick line: $n = 2$. Parameters are: $P_0 = 100$ hPa, $t = 20$ s, $z = -h = -150$ m.

hydroacoustic waves travel faster and are followed by a tail of increasingly longer waves. This is opposite to the dispersion mechanism of transient surface gravity waves, for which longer waves travel faster and are followed by a tail of shorter waves (see Mei *et al*., 2005; Chapter 2). Figure 2 also shows that the hydroacoustic modes have already travelled a very large distance since the onset of motion at $t = 0$, which makes them particularly good candidates for the early detection of incoming surface perturbations generated on the free surface. A practical application of expression (13) to the case of a distributed surface pressure disturbance will be shown at the Workshop.

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