ABSTRACT for IWWWFB 2014

Interference effects on farfield ship waves at high Froude numbers

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Introduction and summary of main result

The Kelvin wake of a ship of length L_s that advances at constant speed V_s along a straight path in calm water is considered. Basic features of this wake have been explained by Kelvin and are well known. Briefly, Kelvin's farfield analysis shows that no ship waves can exist outside a wedge, with angles $\psi \approx \pm 19°28'$ from the ship track, that trails the ship, and that two waves — known as transverse and divergent waves — exist at any point within that wedge. Furthermore, it is also well known that the waves created by a ship mostly stem from the ship bow and stern, where the geometry of a ship hull (a streamlined slender body) varies most rapidly, and that interference between the waves created by these two main wave generators, i.e. the 'bow wave' and 'the stern wave', result in wave amplitudes that vary within the Kelvin wake. These interference effects are well understood, and well known to naval architects. In particular, interference between the *transverse* waves generated at a ship bow and stern results in oscillations ('humps' and 'hollows') in the variation of the wave drag with respect to the ship speed, and avoidance of unfavorable interferences (humps of the wave drag) is an important basic element of ship design that is known to every naval architect.

Interference between the *divergent* waves generated at a ship bow and stern can also occur, although only if the Froude number $F \equiv V_s/\sqrt{gL_s}$ is sufficiently large. Specifically, interference between divergent bow and stern waves can occur in the 'high Froude number regime' $F_c < F$ with $F_c \approx 0.6$. This interference between divergent bow and stern waves results in dominant waves, i.e. largest waves, along rays at angles $\pm\psi$ from the ship track $(x < 0, y = 0, z = 0)$ defined in terms of the Froude number F as

$$\tan\psi = \frac{\sqrt{\pi^2 F^4/\ell^2 - 1}}{2\pi^2 F^4/\ell^2 - 1} \;\text{ with }\; \tan\psi \le \frac{1/2}{\sqrt{2}} \tag{1}$$

Here, $\ell = \xi^{bow} - \xi^{stern}$ denotes the distance between the effective origins $x = \xi^{bow}$ and $x = \xi^{stern}$ of the bow wave and stern waves; ℓ and x are nondimensional in terms of the ship length L_s. Observations show that the effective origins $x = \xi^{bow}$ and $x = \xi^{stern}$ are located slightly aft of the ship bow or slightly ahead of the ship stern, and the empirical distance

$$\ell = \xi^{bow} - \xi^{stern} \approx 0.9 \tag{2}$$

is a reasonable value that is commonly used by naval architects. The relation (1) yields $\tan^2\psi = 1/8$ and $\psi \approx 19°28'$ for $F = F_c$ with

$$F_c = (3/2)^{1/4}\sqrt{\ell/\pi} \approx 0.6 \tag{3}$$

Thus, the largest waves are found along the cusp of the Kelvin wake for $F \approx 0.6$ but inside the Kelvin wake for larger Froude numbers. E.g., (1) yields $\psi \approx 8°12'$ for $F = 1$. Expression (1) yields the approximation

$$\tan\psi \approx \frac{\ell/2}{\pi F^2} \;\text{ as }\; F \to \infty \tag{4}$$

which shows that $\psi = O(1/F^2)$ as $F \to \infty$.

Comparison with recently-published alternative results

The alternative expression

$$\tan\psi = \frac{\sqrt{2\pi F^2 - 1}}{4\pi F^2 - 1} \;\text{ with }\; \tan\psi \le \frac{1/2}{\sqrt{2}} \tag{5}$$

for the 'wake angle' ψ is given in [1] and [2]. This expression yields $\tan^2\psi = 1/8$ for $F = F_c$ with

$$F_c = \sqrt{3}/\sqrt{4\pi} \approx 0.5 \tag{6}$$

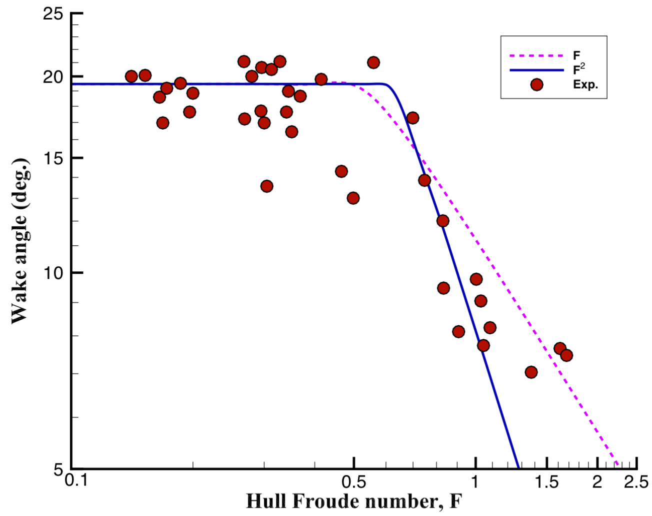

Figure 1: Experimental observations (marked as dots) of the 'wake angle' ψ reported in [1] and corresponding theoretical predictions given by expressions (5) or (1) obtained in [1,2] or from an analysis of interference effects between the divergent waves created by a ship bow and stern, respectively. The curves associated with expressions (5) or (1), which predict that ψ is $O(1/F)$ or $O(1/F^2)$ as $F \to \infty$, are identified as F or F^2, respectively.

Thus, (5) predicts that the largest waves are found along the cusp of the Kelvin wake for $F \approx 0.5$ but inside the Kelvin wake for $0.5 < F$, rather than for $0.6 < F$ in (3). For $F = 1$, (5) yields $\psi \approx 11°23'$. Furthermore, (5) yields the approximation

$$\tan\psi \approx \frac{1/2}{\sqrt{2\pi}F} \quad \text{as} \quad F \to \infty \tag{7}$$

Thus, (5) predicts that ψ is $O(1/F)$ as $F \to \infty$, whereas (4) yields $\psi = O(1/F^2)$ as $F \to \infty$.

Expression (1) is a straightforward consequence of the interference between the divergent waves that are generated at a ship bow and stern, as shown further on. The alternative expression (5) is obtained in [1] by invoking a flow model that assumes that a ship hull of length L_s cannot generate waves of length appreciably larger than L_s, although the wavelength $2\pi V_s^2/g$ of the longest waves created by a ship is larger than the ship length L_s if the Froude number F is greater than $1/\sqrt{2\pi} \approx 0.4$.

Expression (5) is also validated in [1] using numerical computations, and in [2] using both numerical computations and a classical mathematical analysis, of the Kelvin wake generated by a Gaussian distribution of pressure at the free surface. However, such a pressure distribution cannot account for interference effects between the wave systems created by a ship bow and stern, although it may be a reasonable model for the flow around a high-speed boat in the planing regime. Discrepancies between the results obtained in [1,2] for a Gaussian pressure distribution (a wave generator with a single peak) at the free surface and given by an analysis of interference effects between the two dominant waves generated by a ship bow and stern (a 'two-point' wave generator) are therefore not surprising.

Comparison with experimental observations

Fig.1 depicts the theoretical predictions of the 'wake angle' ψ given by expressions (5) or (1) obtained in [1,2] or from an elementary analysis of interference effects between the divergent waves created by a ship bow and stern, respectively. The curves associated with expressions (5) or (1), which predict that ψ is $O(1/F)$ or $O(1/F^2)$ as $F \to \infty$, are identified as F or F^2, respectively, in Fig.1. The experimental observations reported in [1] are also included in Fig.1, where they are marked as dots.

The figure shows that for $F < 0.6$, experimental observations are within the range $13° \leq \psi \leq 21°$ and are mostly located around the Kelvin angle $\psi \approx 19°28'$. Values of ψ that are smaller than the Kelvin angle are expected as a result of interference between the *transverse* waves generated at a ship bow and stern. The experimental observations within the range $0.6 \le F \le 1.1$ are in excellent agreement with the theoretical predictions (1) given by an analysis of interference effects between the *divergent* waves created by a ship bow and stern. The well-established basis of the analysis of interference effects that underlie (1) and the excellent agreement between the theoretical predictions given by (1) and the experimental observations reported in [1] and shown in Fig.1 provide strong evidence that the 'wake angle' ψ along which the largest waves created by a ship are found is merely the result of interference between the divergent waves generated by two main wave generators, the ship bow and stern, where the geometry of a ship hull varies most rapidly.

The three experimental observations in the range $1.3 < F$ are significantly higher than the 'wave-interference' prediction (1) and in good agreement with the 'Gaussian free-surface pressure-distribution' prediction (5). This finding is not surprising because interference between bow and stern waves is much less effective for a high-speed boat in the planing regime than for a displacement ship. Indeed, the flow around a boat in the planing regime can reasonably be represented by means of a free-surface distribution of pressure that has a *single* peak, but a pressure distribution with two separate peaks is required to realistically model the flow about a displacement ship. An elementary analysis can indeed be performed to compare the 'wake angles' ψ produced by single-point or two-point wave generators, and this analysis will be reported elsewhere.

Elementary analysis of interference between bow and stern waves

Expression (1) is now briefly justified (A more complete justification will be given elsewhere). Steady ship waves are a class of dispersive waves governed by the dispersion relation

$$1 \le F^2 k = 1/\cos^2\gamma \equiv 1 + \tan^2\gamma \;\text{ with }\; -\pi \le \gamma \le \pi \tag{8}$$

This dispersion relation defines two distinct dispersion curves, that are symmetric with respect to both the axis $\beta = 0$ and the axis $\alpha = 0$, in the Fourier plane $(\alpha, \beta) \equiv k(\cos\gamma, \sin\gamma)$. The wavelength $\lambda \equiv 2\pi/k$ of the wave generated at the point (8) of a dispersion curve is given by

$$\lambda = 2\pi F^2 \cos^2\gamma \le 2\pi F^2 \equiv \lambda^{max} \tag{9}$$

This expression shows that the longest waves generated by a ship are shorter than the ship length for Froude numbers $F < 1/\sqrt{2\pi} \approx 0.4$ but are longer than the ship length if $0.4 < F$. The Kelvin wake is determined from the dispersion function $\Delta = F^2\alpha^2 - k$, as shown in e.g. [3], and is given by

$$\begin{Bmatrix} -x_n \\ y_n \end{Bmatrix} = 2n\pi F^2 |\cos\gamma| \begin{Bmatrix} 1+\sin^2\gamma \\ \sin\gamma\cos\gamma \end{Bmatrix} \;\text{ with }\; -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \;\text{ and }\; n = 1, 2, 3, \ldots \tag{10}$$

The branch $\pi/2 < \gamma < 3\pi/2$ of the two dispersion curves yields the same wave pattern as the branch $-\pi/2 < \gamma < \pi/2$, which is then only considered in (10). The parametric equations (10) yield $x_n < 0$. This result means that a ship that advances in calm water only generates waves in the region $x < 0$, i.e. behind the ship. Furthermore, the wave pattern (10) is symmetric about the ship's path $y = 0$. Thus, only the positive half $0 \le \gamma < \pi/2$ of the Kelvin wake is now considered.

Farfield waves stem from points of the dispersion curves where the phase $\theta \equiv \alpha x + \beta y$ is stationary. The dispersion relation (8) define two points of stationary phase, which generate two distinct systems of waves known as transverse and divergent waves. The waves generated at a point of stationary phase are radiated in a direction, defined by the angle ψ, that is normal to the dispersion curve and colinear with the group velocity $\mathbf{v}_g \equiv (v_g^x, v_g^y)$ as shown in e.g. [3]. Thus, one has

$$\tan\psi \equiv \frac{y_n}{-x_n} = \frac{v_g^y}{-v_g^x} = \frac{-\Delta_\beta}{\Delta_\alpha} = \frac{\beta/\alpha}{2F^2k-1} = \frac{\sin\gamma\cos\gamma}{1+\sin^2\gamma} = \frac{\tan\gamma}{1+2\tan^2\gamma} = \frac{\tan\varphi}{2+\tan^2\varphi} \tag{11}$$

Here, φ denotes the angle between the ship track and the wave crests. The relations (11) determine the ray along which wave energy is radiated, in the direction of the group velocity $\mathbf{v}_g$, from a point

$$F^2(\alpha, \beta) \equiv F^2 k(\cos\gamma, \sin\gamma) = (\cos\gamma, \sin\gamma)/\cos^2\gamma \tag{12}$$

of the two dispersion curves determined by (8). For short divergent waves, the relations (11) yield

$$2\psi \sim \varphi \sim 1/(\pi/2 - \gamma) \;\text{ as }\; \psi \to 0\;,\;\; \varphi \to 0\;,\;\; \gamma \to \pi/2 \tag{13}$$

The Kelvin wave pattern (10) has a cusp along the ray $\tan^2\psi = 1/8$ where one has $\tan^2\gamma = 1/2$ and $\tan^2\varphi = 2$, i.e. $\psi \approx 19°28'$, $\gamma \approx 35°16'$, $\varphi \approx 54°44'$.

The projection of the distance ℓ between the effective origins of the dominant waves created by a ship bow and stern on the direction of propagation of the waves radiated from the point (12) is $\ell\cos\gamma$ with ℓ given by (2). Unfavorable interference between the waves created by a ship bow and stern, and therefore higher waves and wave drag, occur if the distance $\ell\cos\gamma$ is equal to $\lambda/2$ or more generally if

$$(2n-1)\lambda/2 \equiv (2n-1)\pi F^2\cos^2\gamma = \ell\cos\gamma \;\text{ with }\; 1 \leq n \tag{14}$$

Here, (9) was used. The relation (14) yields

$$1/\cos\gamma = (2n-1)\pi F^2/\ell \;\text{ and }\; \tan^2\gamma = (2n-1)^2\pi^2F^4/\ell^2 - 1 \tag{15}$$

Transverse and divergent waves correspond to $0 \leq \tan^2\gamma \leq 1/2$ and $1/2 \leq \tan^2\gamma$, respectively. Thus, unfavorable interferences between *transverse* waves generated at a ship bow and stern occur if

$$1 \leq \sqrt{(2n-1)\pi/\ell}\,F \leq (3/2)^{1/4} \approx 1.107 \tag{16}$$

The values 1 and $(3/2)^{1/4} \approx 1.107$ are not significantly different. The relation (16) then defines a series of unfavorable Froude numbers that should be avoided. In practice, naval architects use a refined formula that is based on the distance between the crests of the bow and stern waves and a semi-empirical correction for the hull form (prismatic coefficient) to design ships with favorable interference between the transverse waves created by a ship bow and stern. Thus, these interference effects are well understood by naval architects and indeed are an important basic element of ship design. The relations (15) and $1/2 \leq \tan^2\gamma$ show that unfavorable interference between the *divergent* waves created by a ship bow and stern can similarly occur if

$$(3/2)^{1/4}\sqrt{\ell/\pi} \approx 0.6 \leq F \leq \sqrt{2n-1}\,F \tag{17}$$

Furthermore, the relations (15) and (11) show that the largest divergent wave generated by a ship in the high Froude number range $0.6 \leq F$ are found, for $n = 1$, along the ray defined by (1).

Conclusion

The excellent agreement between the theoretical predictions given by (1) and the experimental observations reported in [1] and shown in Fig.1, as well as the well-established wave-interference analysis that underlie expression (1), provide strong evidence that the 'wake angle' ψ along which the largest waves created by a ship are found is merely the unsurprising result of interference between the divergent waves generated by two main wave generators — the ship bow and stern — where the geometry of a ship hull varies most rapidly. Such interference effects are well known in ship hydrodynamics, as well as in other fields that involve waves, but yet are not considered in [1,2]. Expressions (8)-(17) are basic results that are widely known in ship hydrodynamics and naval architecture.

References

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[3] Noblesse, F. & Yang, C. 2007 Elementary water waves, J Engineering Mathematics 59:277-299