Determination of the Wave Resistance of a Towed Body by the Parameters of Generated Waves

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Highlights:

In this work we have obtained exact analytical formulae expressing the wave resistance of a two-dimensional body that moves horizontally at constant speed in a channel of finite depth by the parameters of the downstream waves.

1 Introduction

Consider a two-dimensional body that moves horizontally from right to left at constant speed c in a channel of finite depth h. Assume that in the body frame of reference the flow is steady. Then the wave train generated by the body also moves from right to left with the same velocity c. In the body frame of reference we introduce Cartesian coordinates with the x-axis lying on the bottom and the y-axis directed vertically upward. In this coordinate system far upstream we have a uniform stream with velocity c and far downstream the train of steady periodic waves (Fig.1).

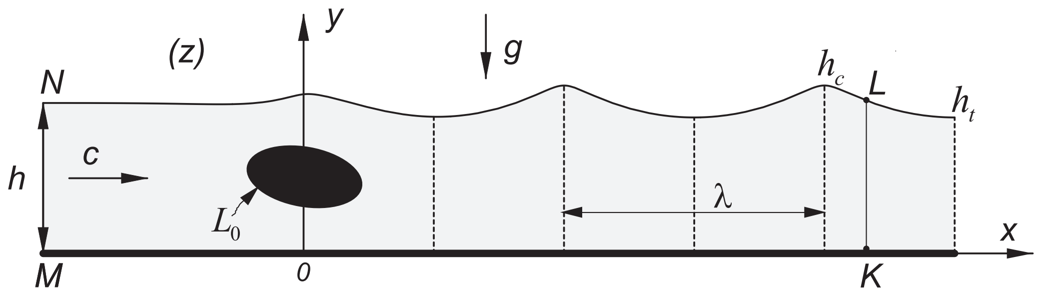

Figure 1: Scheme of the steady flow

Due to the generation of waves the body experiences a resistance. In two-dimensional case the determination of this resistance from properties of the wave train has been the subject of several investigations. The linear theory was first presented by Lord Kelvin [1]. He derived that the wave resistance

$$R_x = \frac{1}{4}\rho g a^2 \left[1 - \frac{4\pi h/\lambda}{\sinh(4\pi h/\lambda)}\right], \tag{1}$$

where ρ is the fluid density, g is the acceleration of gravity, λ is the wavelength and $a = (h_c - h_t)/2$ is the wave amplitude (one half the vertical distance from the crest to the trough). For deep water this formula simplifies and takes the form

$$R_x = \frac{1}{4}\rho g a^2. \tag{2}$$

Wehausen and Laitone [2] have derived an exact resistance formula in terms of the vertical distribution of velocity in the waves and downstream surface hight profile. This formula was later evaluated by Salvesen and von Kerczek [3] using third-order Stokes wave theory.

Duncan [4] was the first to notice that if the properties of the nonlinear wave train far downstream are known, then it is possible to evaluate the wave resistance exactly. Considering the infinite depth case and using the horizontal-momentum equation and the results by Longuet-Higgins [5] he deduced that

$$R_x = cI + 3V - 4T, \tag{3}$$

where I is the mean impulse, V and T are the mean potential and kinetic energies of downstream waves and all these quantities being averaged per unit wave area. Numerical results of [5] and formula (3) allowed Duncan to obtain accurate values of the wave resistance for arbitrary wave steepness.

Another important result of [4] is the formula

$$R_x = \frac{1}{4}\rho g a^2 \left[1 - \frac{3}{2}\frac{(2\pi a)^2}{\lambda^2}\right] \tag{4}$$

that generalizes that of (2) up to the forth power of the amplitude a. The equation (4) has been deduced by making use of the mentioned above result of Wehausen and Laitone [2] and third-order Stokes wave theory.

The formulae (3) is correct only for deep water. The main goal of the presented work is to generalize (3) for water of finite depth and to obtain numerically accurate values of R_x for any amplitude and depth. Besides, using the Stokes method for computing waves on water of finite depth we present an effective algorithm that allows one to generate analytical formulae for the wave resistance with any asymptotic accuracy.

2 Formulae for the wave resistance

Let the shape of the free surface be defined by the equation $y = y_s(x)$. We denote by $\varphi(x, y)$ the potential of the steady flow, then $v_x = \frac{\partial\varphi}{\partial x}$, $v_y = \frac{\partial\varphi}{\partial y}$ are the components and $v = \sqrt{v_x^2 + v_y^2}$ is the modulus of the velocity vector. Far downstream we introduce the following waves properties:
the increment of the potential in the wave

$$\Delta\varphi = \varphi(x + \lambda, y) - \varphi(x, y);$$

the average fluid velocity at any horizontal level completely within the fluid

$$c_a = \frac{1}{\lambda}\int\limits_x^{x+\lambda} v_x(x, y)dx = \frac{\Delta\varphi}{\lambda};$$

the mean depth

$$D = \frac{1}{\lambda}\int\limits_x^{x+\lambda} y_s(x)dx.$$

A usual assumption in the theory of nonlinear periodic waves is that the phase velocity equals the average fluid velocity c_a in the steady motion (see e.g.[5]), then in the unsteady motion the average fluid particle velocity vanishes. In this work we prove that for the waves generated by the moving body this assumption is not correct, i. e. $c \neq c_a$. Moreover we strictly prove that the mean depths far upstream and far downstream are not equal, so there exists a defect of levels $\Delta h = h - D > 0$.

Generally speaking, for the waves on water of finite depth there exist no physical conditions for finding the phase velocity. So, a general form for the mean kinetic energy T of waves which propagate with the phase velocity c_w is

$$T(c_w) = \frac{\rho}{2\lambda}\int\limits_x^{x+\lambda} dx \int\limits_0^{y_s(x)} [(v_x - c_w)^2 + v_y^2]dy.$$

The general form for the mean squared bottom velocity in such an unsteady motion is

$$\sigma^2(c_w) = \frac{1}{\lambda}\int\limits_x^{x+\lambda} [v_x(x, 0) - c_w]^2 dx.$$

The potential energy V we define in an ordinary manner

$$V = \frac{\rho g}{2\lambda}\int\limits_x^{x+\lambda} [y_s(x) - D]^2 dx.$$

By making use of results of Benjamin & Lighthill [6] and Longuet-Higgins [5] we establish that the wave resistance R_x of any body can be found by either of two equivalent formulae:

$$R_x = 3V + \frac{3\rho g}{2}(\Delta h)^2 - \rho(gh - c^2)\Delta h, \tag{5}$$

$$R_x = 3V + \frac{3\rho g}{2}(\Delta h)^2 - 2T(c) - \frac{\rho}{2}\sigma^2(c)h. \tag{6}$$

Consider the case of infinite depth studied by Duncan [4]. For this case $\Delta h \to 0$, $c \to c_a$, $\sigma^2(c)h \to 0$ and the formula (6) takes a simple form

$$R_x = 3V - 2T. \tag{7}$$

Taking into account the Levi-Civita result $2T = cI$ (see e.g. [5]), one can see that (3) is equivalent to (7).

For numerical computations it is convenient to use (5). Equation (5) includes four parameters: the potential energy V, the upstream level h, the defect of levels $\Delta H = h - D$ and the velocity of body c. Computing the potential energy V and the mean level D is usual in the nonlinear wave theory, the question is how to determine h and c. Here the results of Keady & Norbury [7] and Benjamin [8] turn out to be very helpful. Indeed, it follows from the Bernoulli equation and the mass conservation law that h and c satisfy the relations

$$ch = Q, \tag{8}$$

$$c^2 + 2gh = 2R, \tag{9}$$

where Q is the volume flux and R is the total head (Bernoulli constant).

The system of equations (8), (9) has been thoroughly investigated in [7], [8]. It has been established that the system always has two solutions. For the first solution the Froude number $F = c/\sqrt{gh} < 1$ and for the second one $F = c/\sqrt{gh} > 1$. But from the Benjamin results [8] it is possible to deduce that for the second solution the wave resistance $R_x < 0$. In our computations we reject this nonrealistic case and take into account only the first solution with $F < 1$. We should note that the impossibility of $R_x < 0$ means that the body can generate nonlinear waves only if it moves with subcritical speed $c < \sqrt{gh}$.

3 Numerical results

Accurate numerical results has been obtained by the method developed in Maklakov [9]. The method allows one to compute waves of any steepness with accuracy of 10–11 decimal digits. In Fig. 2 and 3 we demonstrate the dependencies of the defect of levels $\Delta h/\lambda$, the difference between the squared phase speeds c_a^2 and c^2, and the wave resistance coefficient $C_x = \frac{R_x}{\rho g \lambda^2}$ on the geometrical wave parameters D/λ and $2a = \frac{h_c - h_t}{\lambda}$. For every given wave depth D/λ the computations have been carried out by changing the wave steepness $2a$ from zero up to the highest wave.

4 Analytical formulae for the wave resistance

Besides numerical computations we present an effective algorithm that allows one to generate analytical formulae for the wave resistance with any asymptotic accuracy. These formulae generalize those of Kelvin (1) and Duncan (4). To develop the algorithm we have applied the Stokes method for computing waves on water of finite depth. The Stokes coefficients have been found from the chain of quadratic equations equivalent to that deduced by Longuet-Higgins [10]. The chain has been derives from Hamilton's principle. After obtaining the Stokes coefficients we expand the wave drag R_x in the series by the powers of the amplitude a. To improve the convergence we sum the series by using Padé approximants. For the case of infinite depth the result is as follows

$$R_x = \frac{\rho g \lambda^2}{(2\pi)^2} \; \frac{0.25\bar{a}^2 - 2.6072\bar{a}^4 + 9.023\bar{a}^6 - 11.304\bar{a}^8 + 2.9096\bar{a}^{10} + 0.932\bar{a}^{12} + 0.055819\bar{a}^{14}}{1 - 8.9289\bar{a}^2 + 24.282\bar{a}^4 - 19.323\bar{a}^6}, \tag{10}$$

where $\bar{a} = \frac{2\pi a}{\lambda}$. The formulae of this type give almost the same results as those calculated by the numerical method of the work [9]. The graphical discrepancy can be seen only for very steep waves whose steepness is about 0.98% of the limiting value. In Fig. 4 we present the comparison of analytical and accurate numerical computations for the case of deep water.

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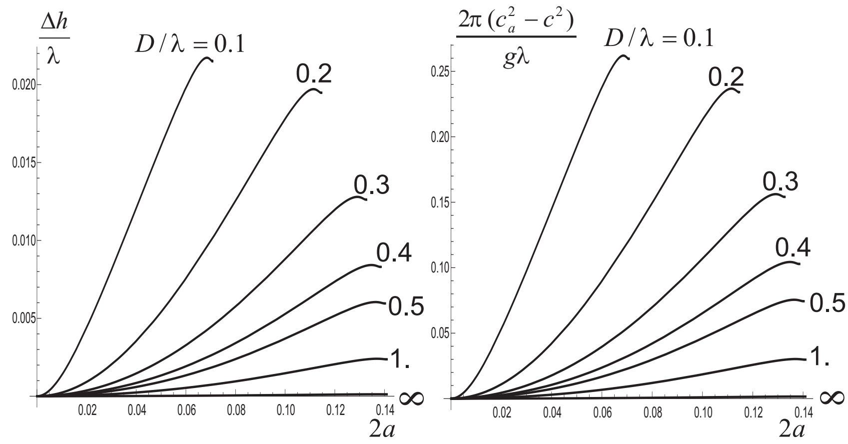

Figure 2: The defect of levels $\Delta h/\lambda$ (on the left) and the difference between the squared phase speeds c_a^2 and c^2 (on the right) versus the wave steepness $2a$.

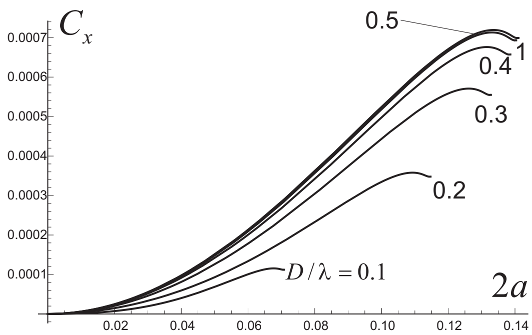

Figure 3: The wave resistance coefficient C_x versus the wave steepness $2a$ for different depths.

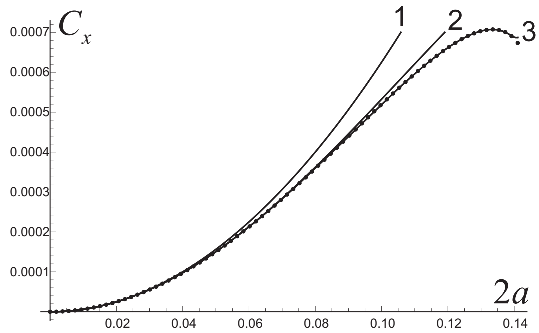

Figure 4: Comparison of analytical and numerical results for deep water: 1 – Kelvin (2), 2 – Duncan (4), 3 – method [9], points – (10).

References

[1] Lord Kelvin. 1887, On Ship Waves. *Proceedings, Institution of Mechanical Engineering*, U.K.

[2] Wehausen, J. V. and Laitone, E. V. 1960, Surface waves. Handbuch der Physics, Vol. 9, Springer, Berline.

[3] Salvesen, N., and von Kerczek, C. 1976, Comparison of Numerical and Perturbation Solutions of Two-Dimensional Nonlinear Water–Wave Problems. *Journal of Ship Research*, Vol. 20, No 3, 160–170.

[4] Duncan, James H. 1983, A note on the Evaluation of the Wave Resistance of Two-Dimensional Bodies from Measurements of the Downstream Wave Profile. *Journal of Ship Research*, Vol. 27, No 2, 90–92.

[5] Longuet-Higgins, M. S. 1975, Integral Properties of Periodic Gravity Waves of Finite Amplitude. *Proc. Roy. Soc. Lond.* A, Vol. 342, 157–174.

[6] Benjamin, T. B. and Lighthill. M. J. 1954, On cnoidal waves and bores. *Proc. Roy. Soc. Lond.* A, Vol. 224, 448–460.

[7] Keady, G. and Norbury J. 1975, Waves and conjugate streams. *J.Fluid Mech.*, Vol. 70, 663-671.

[8] Benjamin, T. B. 1995, Varification of the Benjamin–Lighthill Conjecture About Steady Water Waves *J. Fluid Mech.*, Vol. 295, 337-356.

[9] Maklakov, D. V. 2002, Almost Highest Gravity Waves on Water of Finite Depth. *Euro. Jnl of Applied Mathematics*, Vol. 13, 67-93.

[10] Longuet-Higgins, M. S. 1978, Some New Relations Between Stokes's Coefficients in the Theory of Gravity Waves. *J. Inst. Maths Applies*, Vol. 22, 261-273.