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Development of a highly nonlinear model for wave propagation over a variable bathymetry

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Highlights

- Development of two strategies to model nonlinear waves propagating over variable bathymetry with HOS method.
- Presentation of a validation case to assess accuracy and efficiency of both approaches which are compared.

Introduction

Modeling surface gravity waves is a major concern in ocean engineering, and especially in the field of marine renewable energy. These marine structures (wave and tidal energy converters, offshore wind turbines, ...) are intended to be installed in limited water depth, where the influence of variable bathymetry is very significant on local wave conditions.

In this paper two different schemes for modeling bathymetry in the High-Order Spectral (HOS) method are presented. This highly non-linear potential model has been initially developed by West *et al.* [9] and Dommermuth & Yue [3] for a flat bottom and extensively validated for different configurations. A few HOS studies [5] consider a variable bathymetry, but more has been done in Dirichlet to Neumann Operator method (DNO) [2, 8, 4]. Schäffer [7] proved that HOS and DNO methods are identical thus we propose to characterize these different approaches. The first presented scheme resolves the bottom condition without any approximation whereas the second scheme [5] uses the boundary condition by expanding the surface potential in Taylor series with respect to the mean water depth.

These two schemes are presented here in details and then compared on a validation case considering propagation of nonlinear regular waves. A comparison of their accuracy and efficiency is made with respect to parameters of the problem (steepness, relative water depth, bottom variation ...).

1 Methods and Algorithms

1.1 Hypothesis and formulation of the problem

A 2D rectangular fluid domain with Cartesian coordinate system is considered. The z axis is vertical and oriented upwards, with the level $z=0$ corresponding to the mean water level. $z=\eta(x,t)$ represents the free surface elevation, h the total depth, h_0 the mean depth and β the bottom variation, such as $-h(x)=-h_0+\beta(x)$ (*cf* Fig.1). We assume periodic boundary conditions in the horizontal plane so that the domain is considered infinite.

A potential flow formalism is used (incompressible and inviscid fluid, irrotational flow). Given these assumptions, the velocity field $\vec{V}$ derives from a potential $\vec{V}(x,z,t)=\vec{\nabla}\phi$ and the continuity equation becomes the Laplace equation in the fluid domain:

$$\Delta\phi = 0 \tag{1}$$

Following Zakharov [10], both fully nonlinear free-surface boundary conditions (kinematic and dynamic) are written in terms of surface quantities η and $\widetilde{\phi}$:

$$\frac{\partial\eta}{\partial t} = \left(1+|\nabla\eta|^2\right)\frac{\partial\phi}{\partial z} - \nabla\widetilde{\phi}.\nabla\eta \qquad \text{on } z=\eta(x,t) \tag{2}$$

$$\frac{\partial\widetilde{\phi}}{\partial t} = -g\eta - \frac{1}{2}\left|\nabla\widetilde{\phi}\right|^2 + \frac{1}{2}\left(1+|\nabla\eta|^2\right)\left(\frac{\partial\phi}{\partial z}\right)^2 \qquad \text{on } z=\eta(x,t) \tag{3}$$

with $\widetilde{\phi}(x,t)=\phi(x,z=\eta,t)$ standing for the free surface velocity potential and ∇ the horizontal gradient.

To account for the time evolution of the quantities of interest η and $\widetilde{\phi}$ one only needs to evaluate the vertical velocity at the free surface $W(x,t)=\frac{\partial\phi}{\partial z}(x,z=\eta(x,t),t)$. Note that HOS method was initially developed for a flat bottom, while here the bottom boundary condition reads:

$$\frac{\partial\phi}{\partial x}\frac{\partial\beta}{\partial x} - \frac{\partial\phi}{\partial z} = 0 \qquad \text{on } z=-h_0+\beta(x) \tag{4}$$

To account for a variation of the bathymetry, an additional potential is introduced following [8, 4, 5]. The total potential ϕ_{tot} solution of the wave propagation over variable bottom problem is expressed as $\phi_{tot} = \phi_{h_0} + \phi_\beta$:

- ϕ_{h_0} is solution of the problem at constant depth h_0 and therefore satisfies a Neumann condition on $z = -h_0$: $\frac{\partial \phi_{h_0}}{\partial z}(x, z = -h_0, t) = 0$
- ϕ_β allows the definition of the correct bottom boundary condition (Eq.(4)) and satisfies a Dirichlet condition on $z = 0$: $\phi_\beta(x, z = 0, t) = 0$

These potentials are defined on a specific set of basis functions with k_{j_1} the wavenumbers and A_{j_1}, B_{j_1} the modal amplitudes:

$$\phi_{h_0}(x,z,t) = \sum_{j_1} A_{j_1}(t) \frac{\cosh(k_{j_1}(z+h_0))}{\cosh(k_{j_1} h_0)} e^{ik_{j_1}x} \tag{5}$$

$$\phi_\beta(x,z,t) = \sum_{j_1} B_{j_1}(t) \frac{\sinh(k_{j_1} z)}{\cosh(k_{j_1} h_0)} e^{ik_{j_1}x} \tag{6}$$

1.2 High-Order Spectral Method

The HOS model [9, 3] is based on a pseudo-spectral method where the velocity potential is expressed as a truncated power series of components $\phi^{(m)}$ for $m = 0$ to M (M standing for the order of the method). Then, the potential taken at the free surface is expanded in a Taylor series with respect to the mean water level $z = 0$, *cf* Eq.(7). Combining these two expansions gives a triangular set of Dirichlet problems for the components that can be solved by means of a spectral method, taking advantage of the use of Fast Fourier Transforms (FFTs).

$$\widetilde{\phi}_{tot}(x,t) = \phi_{tot}(x, z = \eta, t) = \sum_{m=1}^{M} \sum_{n=0}^{M-m} \frac{\eta^n}{n!} \frac{\partial^n \phi_{tot}^{(m)}}{\partial z^n}(x, z=0, t) \tag{7}$$

Introducing the two potentials ϕ_{h_0} and ϕ_β, one more equation is needed to obtain their modal amplitudes at each order m : $A_{j_1}^{(m)}(t)$ and $B_{j_1}^{(m)}(t)$ respectively (*cf* Eqs.(5) & (6)). Those potentials are explicitly linked through the bottom boundary condition and two approaches have been developed to close the problem.

Once $A_{j_1}^{(m)}(t)$ and $B_{j_1}^{(m)}(t)$ are computed, the vertical velocity on the free surface $W(x,t)$ can be obtained. The same kind of double expansion is used both for the potential and for its vertical derivative. Then, free surface boundary conditions Eqs.(2) & (3) allows to advance in time the solution of the problem $\widetilde{\phi}_{tot}$ and η.

$$W(x,t) = \frac{\partial \phi_{tot}}{\partial z}(x, z=\eta, t) = \sum_{m=1}^{M} \sum_{k=0}^{m-1} \frac{\eta^k}{k!} \frac{\partial^{k+1} \phi_{tot}^{(m-k)}}{\partial z^{k+1}}(x, 0, t) \tag{8}$$

1.3 Exact method

The first approach used to take into account the bottom boundary condition is to directly substitute the expressions of ϕ_{h_0} and ϕ_β in this condition Eq.(4). One obtains, at each order m:

$$\sum_{j_1} B_{j_1}^{(m)}(t) = \sum_{j_1} A_{j_1}^{(m)}(t) \frac{\left[-\sinh k_{j_1}.\beta(x) + i.\frac{\partial \beta(x)}{\partial x} \cosh k_{j_1}.\beta(x)\right]}{\left[\cosh k_{j_1}(h_0 - \beta(x)) + i.\frac{\partial \beta(x)}{\partial x} \sinh k_{j_1}(h_0 - \beta(x))\right]} \tag{9}$$

In this case, no assumption is made on the magnitude of β and the bottom condition is solved exactly. Nevertheless, one has to note that the modal amplitudes $B_{j_1}^{(m)}(t)$ depend actually on the location x. Consequently, the decomposition Eq.(6) relies on modal amplitudes depending on x: $B_{j_1}^{'(m)}(x,t) = A_{j_1}^{(m)}(t).D(x)$, which prevents from the use of FFTs: this is a direct method. The associated computational cost is expected to be very important especially for 3D simulations, leading to the development of a second method presented hereafter.

1.4 Approximate method

The second approach considered to impose the bottom boundary condition Eq.(4) is the one used in Liu & Yue [5]. Another expansion of ϕ_{tot} is proposed with respect to the mean water depth $z = -h_0$ for this boundary condition, assuming β and $\frac{\partial \beta}{\partial x}$ to be small parameters. This leads to a new triangular system that reads:

$$m = 1 : \frac{\partial \phi_{tot}^{(1)}}{\partial z}(x, -h_0, t) = 0 \tag{10}$$

$$m = 2, ..M : \frac{\partial \phi_{tot}^{(m)}}{\partial z}(x, -h_0, t) = \sum_{l=1}^{m-1} \frac{\partial}{\partial x} \left[\frac{\beta^l}{l!} \frac{\partial^{l-1}}{\partial z^{l-1}} \left(\frac{\partial \phi_{tot}}{\partial x} \right)^{(m-l)} \right]_{z=-h_0} \tag{11}$$

Then, at each order m, this gives the relationship between $B_{j_1}^{(m)}$ and known quantities: $A_{j_1}^{(m)}$ and their previous orders. The main point in this procedure is the possible use of FFT and its interesting computational costs, in contrast with the exact direct method. In the following, the order of truncation of power series (at free surface and bottom) is assumed to be the same, following [5].

2 Validation

2.1 Test case

The main objective of this paper is to assess the domains of applicability of both approaches. The proposed validation case is the simple propagation of a non-linear regular wave over flat bottom (see Fig.1). The non-linear initialization of η and $\widetilde{\phi}$ is provided by the stream function solution of Rienecker & Fenton [6] build for a constant water depth h. The complexity of this test case relies on the definition of the potential ϕ_{h_0} at the 'mean' depth h_0, which differs from h. Then, the second potential ϕ_β has to take care of this difference in water depth, the total potential ϕ_{tot} being the solution of the problem at 'total' water depth $h = h_0 - \beta$. This depth variation β is introduced artificially and is assumed to be constant for this study.

This validation case allows to characterize both approaches in terms of accuracy and computational efficiency. These may also be compared to the initial HOS model which has been validated on such test case [1]. Note that this is intended to be a complicated case since an inadequate depth is imposed in all the domain. Thus, it is assumed that if the results are correct on this complex configuration, the method is validated.

In order to advance in time unknowns on the free surface η and $\widetilde{\phi}$, one has to evaluate the vertical velocity $W(x,t)$. This validation case proposes to study the error $\epsilon_w = \frac{W - W_{RF}}{W_{RF}}$ made on the evaluation of this vertical velocity at the initial time step (W_{RF} represents the reference vertical velocity from Rienecker and Fenton [6]). This is an appropriate measure of the accuracy of the model and it has been shown that if this error is limited, the propagation will take place properly in the model [1].

The regular wave is described by its steepness $k.a$ (k the wavenumber and a the amplitude) and its relative water depth $k.h$. The influence of these wave parameters is investigated as well as the non-dimensional variation of the bottom $\frac{\beta}{h_0}$ and numerical parameters: number of nodes N and HOS order M.

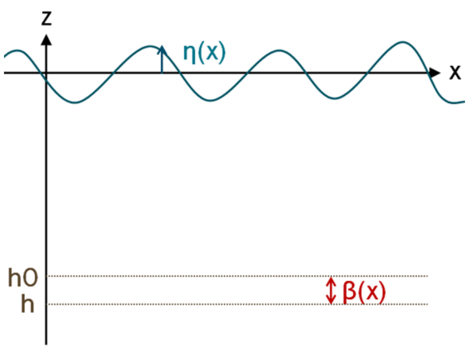

Figure 1: Validation case

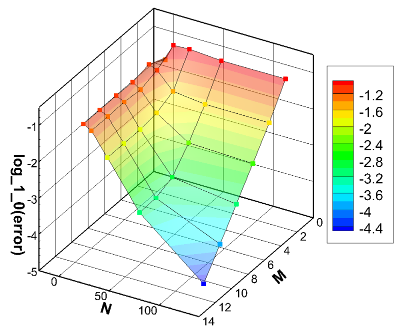

Figure 2: Exact method: $k.h = 0.916$, $k.a = 0.2$, all $\frac{\beta}{h_0}$. Error on W ($log(\epsilon_w)$) as function of N and M.

2.2 Results

2.2.1 Exact method

It appears that the behavior of the exact method is identical to the original HOS model, whatever the variation of the bathymetry imposed $\left(\frac{\beta}{h_0}\right)$. This is due to the fact that one solves the exact bottom boundary condition which allows to keep the convergence properties of the original HOS model as shown in Fig.2: exponential convergence of spectral method with respect to number of point N and HOS order M. Thus, this method allows simulations up to highly non-linear waves and very limited water depth as shown in [1]. However, as indicated before, the computational effort largely increases with this approach, compared to the original HOS scheme.

2.2.2 Approximate method

Contrary to the exact method, the behavior of the approximate one depends strongly on the value of the bottom variation $\frac{\beta}{h_0}$. Figure 3 presents the convergence on W with respect to the number of points N and HOS order M. The original HOS method, *i.e.* $\frac{\beta}{h_0} = 0\%$ and consequent bottom variation $\frac{\beta}{h_0} = 25\%$ are compared. It is interesting to notice that this approach converges for this magnitude of bottom variation, even if, as expected, this convergence is slower than the original method. However, note that the spectral properties (exponential convergence) are conserved.

Different set of parameters have been tested with some results presented in Tab.1 at fixed discretization and HOS order: $N = 64$, $M = 8$. One can see that the error on the vertical velocity depends strongly on the three parameters $k.a$, $k.h$ and $\frac{\beta}{h_0}$. Errors rise with steepness, bottom variation and for decreasing water depth. However, as shown previously, convergence properties are globally conserved in this range of parameters, showing the applicability of this approximate method to treat classical problems (and reminding that the validation case used is very demanding). At the same time, the computational effort is greatly reduced compared to exact method, making large 3D computations accessible.

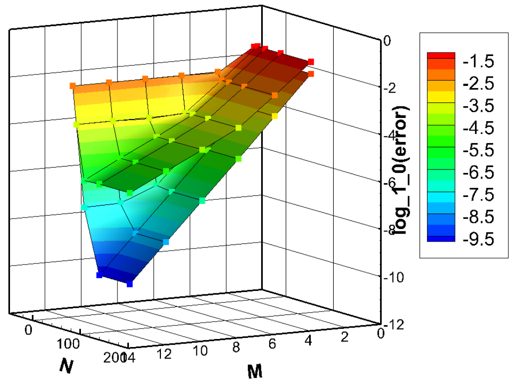

Figure 3: Approximate method: $k.h = 0.916$, $k.a = 0.1$. Influence of $\frac{\beta}{h_0}$ on error on W $(log\,(\epsilon_w))$ as function of N and M. Low-surface: $\frac{\beta}{h_0} = 0\%$, high: $\frac{\beta}{h_0} = 25\%$

	k.h=0.916		
β/h_0	k.a=0.01	k.a=0.05	k.a=0.1
0%	1.7 10^{-14}	4.0 10^{-10}	2.5 10^{-7}
5%	4.8 10^{-12}	1.8 10^{-9}	3.0 10^{-7}
25%	5.5 10^{-8}	2.6 10^{-6}	3.4 10^{-5}
50%	3.9 10^{-6}	1.1 10^{-4}	1.2 10^{-3}
	k.a=0.1		
β/h_0	k.h=0.67	k.h=3	k.h=10
0%	5.7 10^{-8}	6.3 10^{-9}	6.2 10^{-9}
5%	2.2 10^{-7}	6.3 10^{-9}	6.2 10^{-9}
25%	3.9 10^{-5}	1.8 10^{-6}	2.6 10^{-7}
50%	1.1 10^{-3}	3.8 10^{-4}	1.8 10^{-4}

Table 1: Relative error on W for the approximate method, $N = 64$, $M = 8$

When dealing with highly non-linear waves ($k.a = 0.2-0.4$), stability problems will occur for large bottom variation or shallow water. For instance, for $k.h = 0.916$ and $\frac{\beta}{h_0}$ varying from 0% to 25% this method converges for all $k.a$, while for $\frac{\beta}{h_0} = 50\%$, this method doesn't converge for $k.a \geq 0.2$.

A detailed study will be provided at the workshop to establish the exact application range of this approximate method. In addition, an in-depth comparison of computational effort at given accuracy between both exact and approximate method will be provided. Nevertheless, we are confident in the fact that a lot of combinations of ($k.a$, $k.h$, $\frac{\beta}{h_0}$) can be computed, with lower computational time. Finally, a validation case of propagating waves over an underwater bar ([8] [4]) will also be performed and presented at the workshop.

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