Transparency of structures in water waves

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Highlights:

• A proof is given illustrating that structures satisfying the John condition in two-dimensions cannot transmit all incident wave energy.

• Computations performed for submerged steps and horizontal plates illustrate configurations for which incident waves are totally transmitted beyond the structure without a change of phase.

1. Introduction

The term 'cloaking' in the linearised water wave context has been used to describe the process by which a structure subject to small amplitude time-harmonic incident waves is rendered invisible to the observer in the far field by the addition to/alteration of the geometry in a finite region outside that structure. Thus, the cloaked structure is designed to scatter no wave energy in any polar direction and the far-field observer sees only the incident wave field. Porter & Newman (2014) have provided compelling numerical evidence that a vertical circular cylinder extending uniformly throughout the depth can be cloaked using an annular region of variable bathymetry, whilst Newman (2014) has explored the use of circular arrangements of surface-piercing cylinders to cloak a central vertical cylinder.

In this paper our main consideration is the related two-dimensional problem. Now a system of fixed or, perhaps, moving two-dimensional structures is said to be cloaked in small amplitude time harmonic water waves if there is no reflected wave and the amplitude and phase of the transmitted wave is the same as that of the incident wave. That is, the structures appear transparent to the incident wave field when observed in the far-field.

The purpose of this work is two-fold: (i) to provide general conditions under which cloaking cannot be achieved; and (ii) to provide numerical examples of cloaking. Theoretical results will be presented below for an infinite depth fluid but their generalisation to a fluid of constant finite depth is also possible.

2. Statement of the problem

We adopt two-dimensional Cartesian coordinates (x, z) with $z = 0$ in the mean free surface and the fluid extending into $z < 0$. The fluid is incompressible and inviscid and the flow irrotational. Small amplitude waves of a single radian frequency ω are incident from $x = -\infty$. After removing a time-harmonic dependence, the velocity potential $\Phi(x, z)$, used to describe the flow, satisfies

$$\nabla^2\Phi(x, z) = 0, \quad \text{in } \mathcal{D}, \tag{1}$$

the domain occupied by the fluid,

$$\frac{\partial \Phi}{\partial z} - K\Phi = 0, \quad \text{on } \mathcal{F}, \tag{2}$$

the free surface, where $K = \omega^2/g$,

$$\frac{\partial \Phi}{\partial n} = 0, \quad \text{on } \mathcal{B}, \tag{3}$$

the union of wetted surfaces of any rigid bodies in the fluid and

$$|\nabla\Phi| \to 0, \quad \text{as } z \to -\infty. \tag{4}$$

In order that the incident wave is totally transmitted it is required that

$$\Phi(x, z) \sim \begin{cases} e^{iKx+Kz}, & x \to -\infty, \\ Te^{iKx+Kz}, & x \to \infty, \end{cases} \tag{5}$$

where energy considerations imply $|T| = 1$. There are many well-known examples of structures satisfying these transmitting conditions (specifically $|T| = 1$) including wave scattering by submerged circular cylinders, pairs of vertical barriers, submerged horizontal plates and topographical features such as ripple beds and sea mounts, when the finite depth variation of the equations above is included.

For cloaking, however, it is additionally required that $T = 1$. That is, there is no phase shift in the transmitted wave.

3. A non-transmitting result for John bodies

Let the two-dimensional system of bodies satisfy the so-called John condition. That is $\mathcal{B}$ intersects $\mathcal{F}$ in at most 2 points, say $x = \pm a$, $z = 0$ and vertical lines drawn downwards from those two points from the free surface do not intersect any other bodies; see Fig. 1. Let $\mathcal{F}_\mp$ be portions of the free surface given by $\{-M < x < -a\}$ and $\{a < x < M\}$ and $\mathcal{D}_\mp$ the fluid below each free surface. Let $\mathcal{D}_B$ be the remaining fluid

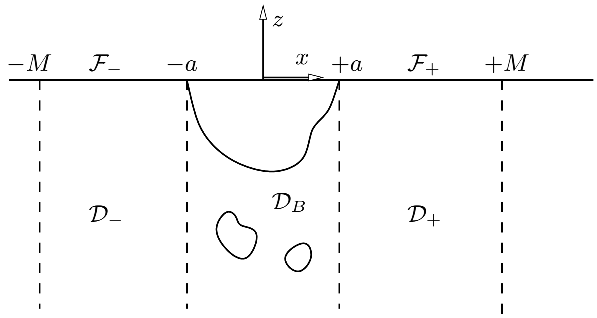

Figure 1: Schematic of a system of bodies satisfying the John condition

region surrounding $\mathcal{B}$ in $-a < x < a$ and such that $\mathcal{D}_B$ is a non-zero region.

Application of Green's first identity to Φ and $\overline{\Phi}$ in $\mathcal{D}_- \cup \mathcal{D}_B \cup \mathcal{D}_+ \subset \mathcal{D}$, before taking the limit $M \to \infty$ and using $|T| = 1$, gives

$$0 = \int_{\mathcal{D}_B} |\nabla\Phi|^2 \mathrm{d}V + \lim_{M\to\infty} \left[\int_{\mathcal{D}_+\cup\mathcal{D}_-} |\nabla\Phi|^2 \mathrm{d}V - K \int_{\mathcal{F}_+\cup\mathcal{F}_-} |\Phi|^2 \mathrm{d}x \right]. \quad (6)$$

By analytic continuation, in order to satisfy (5), Φ cannot be equal to a constant everywhere in $\mathcal{D}_B$ and so the first term in (6) is positive and hence

$$\lim_{M\to\infty} \left[\int_{\mathcal{D}_+\cup\mathcal{D}_-} |\nabla\Phi|^2 \mathrm{d}V - K \int_{\mathcal{F}_+\cup\mathcal{F}_-} |\Phi|^2 \mathrm{d}x \right] < 0. \quad (7)$$

Note the strict inequality, which will be important.

Let $b > a$ and apply Green's second identity to Φ and $\mathrm{e}^{\mathrm{i}K(x-b)+Kz}$ in $x \geq b$, $-\infty < z < 0$. Both functions represent outgoing waves at infinity and it follows that the only contribution comes from the line $x = b$, resulting in

$$\int_{-\infty}^{0} \left[\frac{\partial\Phi}{\partial x} \mathrm{e}^{Kz} - \mathrm{i}K\Phi \mathrm{e}^{Kz} \right]_{x=b} \mathrm{d}z = 0. \quad (8)$$

Integration by parts followed by the use of the Cauchy-Schwarz inequality gives

$$|\Phi(b,0)|^2 \leq \left[\int_{-\infty}^{0} \left| \frac{\partial\Phi}{\partial z} - \mathrm{i}\frac{\partial\Phi}{\partial x} \right|^2_{x=b} \mathrm{d}z \right] \left[\int_{-\infty}^{0} \mathrm{e}^{2Kz} \mathrm{d}z \right]. \quad (9)$$

For two complex numbers A and B, $|A+B|^2 = 2(|A|^2 + |B|^2) - |A-B|^2 \leq 2(|A|^2 + |B|^2)$ and applying this to the first integrand in (9) gives

$$\left| \frac{\partial\Phi}{\partial z} - \mathrm{i}\frac{\partial\Phi}{\partial x} \right|^2 \leq 2|\nabla\Phi|^2. \quad (10)$$

Combining this result with (9) gives

$$K|\Phi(b,0)|^2 \leq \int_{-\infty}^{0} |\nabla\Phi|^2_{x=b} \, \mathrm{d}z \quad (11)$$

and integrating over $a < b < M$ shows that

$$\int_{\mathcal{D}_+} |\nabla\Phi|^2 \mathrm{d}V - K \int_{\mathcal{F}_+} |\Phi|^2 \mathrm{d}x \geq 0. \quad (12)$$

Applying the same arguments as above to the region $x < -a$ starting out with a point $b < -a$ results in a second inequality in the form of (12) but with $\mathcal{D}_-$ and $\mathcal{F}_-$ replacing $\mathcal{D}_+$ and $\mathcal{F}_+$. Summing these two results and letting $M \to \infty$, before comparing with (7) results in a contradiction. Thus the original assumption that $|T| = 1$ cannot hold and it is shown that systems of bodies satisfying the John condition can never be totally transmitting and hence can never be cloaking structures.

The same conclusion can be drawn for a system of bodies satisfying the John condition in constant finite depth using the same type of approach, although there is some additional algebraic complexity.

4. Wide-spacing conditions for cloaking

Consider first a two-dimensional structure which scatters normally-incident waves which is centred around the line $x = 0$ such that far enough to the left (or right) of $x = 0$ the depth is constant and equal to h_1 (or h_2). Let potentials associated with waves incident from $x = -\infty$ and $x = +\infty$ be denoted by $\Psi_1(x,z)$ and $\Psi_2(x,z)$. Then, in the far field,

$$\Psi_1(x,z) \sim \begin{cases} t_1 \mathrm{e}^{\mathrm{i}k_2 x} \psi_2(z), & x \to \infty \\ (\mathrm{e}^{\mathrm{i}k_1 x} + r_1 \mathrm{e}^{-\mathrm{i}k_1 x}) \psi_1(z), & x \to -\infty \end{cases}$$

and

$$\Psi_2(x,z) \sim \begin{cases} (\mathrm{e}^{-\mathrm{i}k_2 x} + r_2 \mathrm{e}^{\mathrm{i}k_2 x}) \psi_2(z), & x \to \infty \\ t_2 \mathrm{e}^{-\mathrm{i}k_1 x} \psi_1(z), & x \to -\infty \end{cases}$$

where r_i and t_i are complex reflection and transmission coefficients, k_i satisfies the dispersion relation $K = k_i \tanh k_i h_i$ and

$$\psi_i(z) = N_i^{-1/2} \cosh k_i(z + h_i) \quad (13)$$

where $N_i = \frac{1}{2}(1 + \sinh(2k_i h_i)/(2k_i h_i))$. Then it is well known (e.g. Newman (1965)) that

$$(k_1 h_1) t_2 = (k_2 h_2) t_1, \qquad \arg(t_1) = \arg(t_2) \equiv \delta, \quad (14)$$

say, and

$$\delta_1 + \delta_2 = 2\delta \pm \pi, \quad \text{where} \quad r_i = |r| \mathrm{e}^{\mathrm{i}\delta_i} \quad (15)$$

in addition to $|r|^2 + |t_1 t_2| = 1$.

Imagine now an opposing pair of such scatterers which are centred around $x = \pm b$ and symmetric in the line $x = 0$. Far to the right/left of the scatterer at $x = \pm b$ the fluid is of constant depth h_1 and in between the two scatterers the fluid is of constant depth h_2. On the assumption that $k_2 b \gg 1$ and evanescent wave interactions between the two scatterers are negligible, the wide-spacing approximation (Newman (1965)) can be used to give the transmission coefficient, T, for waves on the complete arrangement given by

$$\begin{aligned} T &= \frac{t_1 t_2 e^{2i(k_2-k_1)b}}{1 - r_2^2 e^{4ik_2 b}} \\ &= \frac{(1-|r|^2)e^{2i(\delta+(k_2-k_1)b)}}{1-|r|^2 e^{2i(\delta_2+2k_2 b)}} \end{aligned} \quad (16)$$

after using the relations above. It follows that for cloaking ($T = 1$) two conditions are required:

$$\left.\begin{aligned} \delta_2 + 2k_2 b = n\pi \\ \delta + (k_2 - k_1)b = m\pi \end{aligned}\right\} \quad n, m \in \mathbb{Z}. \quad (17)$$

For transmitting ($|T| = 1$) only the first condition is needed.

5. Numerical examples of cloaking

5.1 A submerged horizontal plate

We consider a thin horizontal plate of length $2b$ submerged to a depth h_2 in water of depth h_1. According to the wide-spacing theory of §4, we require only the phases of the reflection and transmission coefficients for a submerged semi-infinite plate with one end at $x = 0$. This problem has an exact solution which is found, for example, using the Wiener-Hopf technique (e.g. as in Linton & McIver (2001)) and results in

$$\delta_2 = -2f(k_2) \pm \pi, \quad \delta = f(k_1) - f(k_2) \pm \pi \quad (18)$$

where

$$\begin{aligned} f(s) &= (s/\pi)(\eta \ln(\eta/h_1) + h_2 \ln(h_2/h_1)) - \tfrac{1}{2}\pi \\ &\quad + \sum_{n=1}^{\infty} \big(\tan^{-1}(s/k_{2,n}) - \tan^{-1}(s/k_{1,n}) \\ &\qquad + \tan^{-1}(s\eta/n\pi) \big) \end{aligned} \quad (19)$$

where $\eta = h_1 - h_2$, and $k_{i,n}$ are positive roots of $K = -k_{i,n} \tan k_{i,n} h_i$.

Exact results for the scattering of waves by a submerged horizontal plate of length $2b$ in either finite or infinite depth can be found using a number of methods, all eventually requiring numerical computation. Here, we have implemented a new method based taking Fourier transforms in x which leads to an an integro-differential equation for the unknown jump in the potential $P(x) = \Phi(x, -h_2^-) - \Phi(x, -h_2^+)$, say, across the plate expressed as

$$\begin{aligned} k_1 e^{ik_1 x} &= \frac{1}{2\pi} \frac{d^2}{dx^2} \int_{-b}^{b} P(x') \log|x - x'| dx' \\ &\quad + \frac{1}{2\pi} \fint_{-\infty}^{\infty} l^2 E(l) e^{ilx} \int_{-b}^{b} P(x') e^{-ilx'} dx' dl \end{aligned} \quad (20)$$

for $|x| < b$ in terms of a principal-value integral, with

$$E(l) = \frac{2\sinh(l\eta)(l \sinh lh_2 - K \cosh lh_2)}{l(l \sinh lh_1 - K \cosh lh_1)} - \frac{1}{|l|}$$

having a pole at $l = k_1$. An approximation to (20) is implemented, first by expanding

$$P(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x/b), \qquad |x| < b \quad (21)$$

where α_n are coefficients to find, with

$$p_n(t) = \frac{e^{in\pi/2}}{\pi(n+1)} (1 - t^2)^{1/2} U_n(t) \quad (22)$$

defined in terms of second-kind Chebychev polynomials, $U_n(t)$, and incorporating the expected square-root behaviour as the ends of the plate are approached. Using (21) in (23) before multiplying through by $p_m^*(x/b)$ and integrating over $-b < x < b$, a process characterising the Galerkin method, leads to the real algebraic system of equations

$$\frac{\alpha_m}{2\pi(m+1)} + \sum_{n=0}^{\infty} \alpha_n K_{mn} = F_m, \quad m = 0, 1, \ldots \quad (23)$$

where

$$F_m = J_{m+1}(k_1 b) \quad (24)$$

and

$$K_{mn} = \frac{1}{2\pi} \fint_{-\infty}^{\infty} E(l) J_{m+1}(lb) J_{n+1}(lb)\, dl. \quad (25)$$

We note that: (i) $E(l) \sim e^{-2|l|\min\{h_2,\eta\}}$ as $|l| \to \infty$ and so the integrals (25) are rapidly convergent; (ii) $K_{mn} = K_{nm}$ and $K_{mn} = 0$ if $m+n$ is odd which allows (23) to be decoupled into even and odd systems; (iii) the solution does not require the roots, $k_{i,n}$, $n \geq 1$ of the dispersion relation.

The conditions for cloaking turn out to require

$$\sum_{n=0}^{\infty} \alpha_{2n} F_{2n} = \sum_{n=0}^{\infty} \alpha_{2n+1} F_{2n+1} = 0. \quad (26)$$

Numerical solutions of (23) converge rapidly with increasing truncation size (dependent on wavenumber) and typically fewer than 10 terms are sufficient to guarantee 6 decimal place accuracy. Results for infinite depth are most easily found by explicitly taking the limit $h_1 \to \infty$.

The conditions for cloaking are shown in Fig. 2(a,b). Wide-spacing computations satisfying (17) with (18) are given by dots, whilst lines depict exact results satisfying (26). The agreement between the two is generally good although the wide-spacing approach neglects interactions underneath the plate which can have a significant effect. The plots show the variation of (a) h_2/h_1 and (b) $k_2 b$ both with b/h_2, the dimensionless length of the plate. The filled-in circles show the exact infinite depth results. Families of results emerge, each family associated with a number of wavelengths over the plate, characterised here by $k_2 b$.

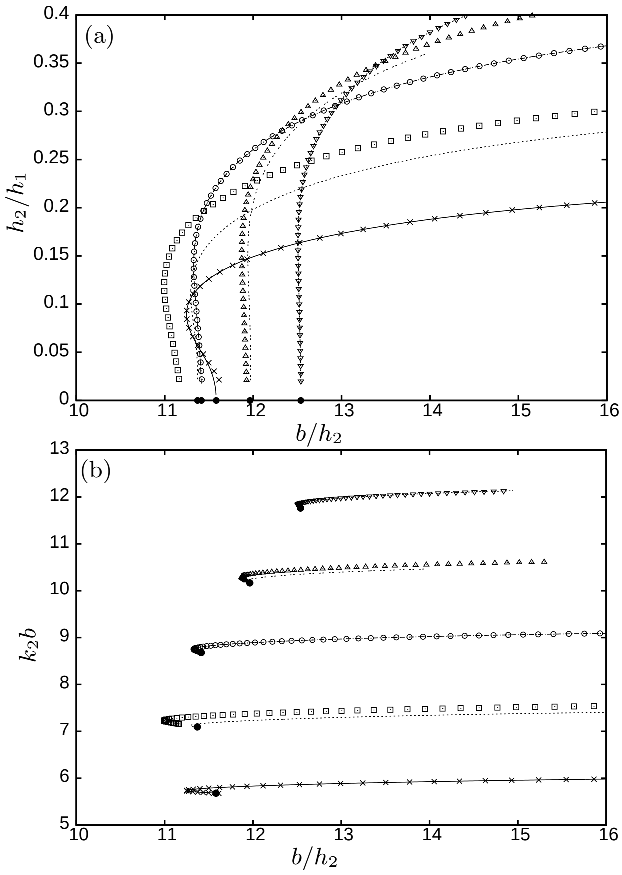

Figure 2: Parameter variations for transparency of a submerged plate. Lines and symbols show exact and wide-spacing results.

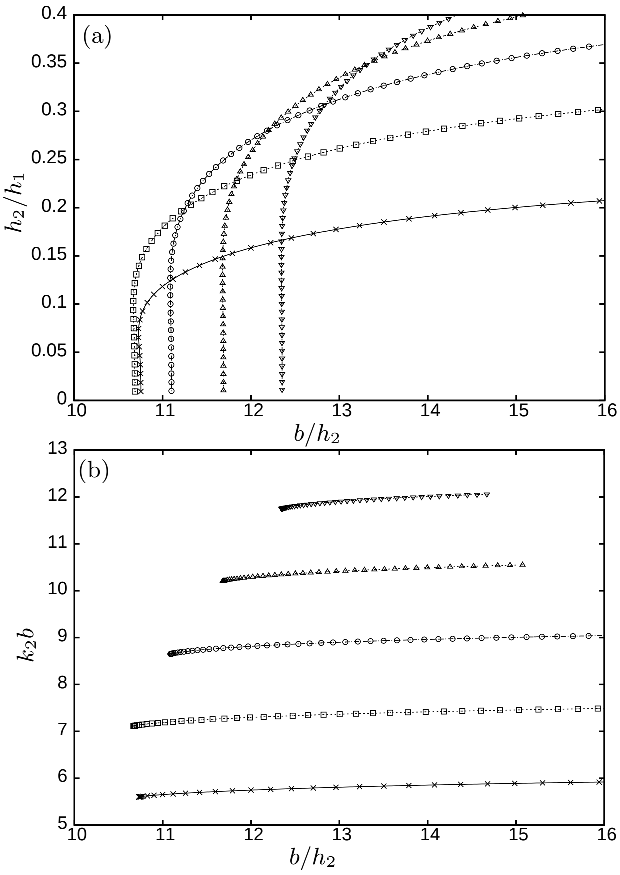

Figure 3: Parameter variations for transparency of a submerged step. Lines and symbols show exact and wide-spacing results.

5.2 A submerged step

Using integral equation methods outlined in Porter (1995), we have also considered wide-spacing and exact results for cloaking of a rectangular step of length $2b$ rising from a depth h_1 to a depth h_2. Cloaking results are shown for the step in Fig. 3(a,b) in the same format as used in Fig. 2(a,b). As the geometry of the step is similar to that of the submerged horizontal plate, it is unsurprising that the two sets of results are similar. However, in this example the wide-spacing approximation is always in extremely close agreement with the exact results as all fluid interactions take place over the step here.

6. Summary and further work

The numerical results suggest that transparency, or cloaking, is only possible if the structure is sufficiently long compared to its submergence and for high enough frequencies (here $b/h_2 > 10$ and $k_2b > 5$). Other results to be shown at the workshop will include the effect of breaking the symmetry and, perhaps, three-dimensional extensions. Efforts to extend the proof of the John condition to three-dimensions and produce existence results in more general settings are ongoing.

References

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