Modified shallow water equations for mild-slope seabeds

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Highlights: Modified shallow water equations, Bathymetry effects, Variational formulation, Hyperbolic structure, Tsunamis modelling.

1 Introduction

The celebrated classical nonlinear shallow water (Saint-Venant) equations were derived in the nineteenth century [13]. These equations are still widely used in practice and the literature counts thousands of publications devoted to the applications, validations or numerical solutions of these equations. Some important attempts have been also made to improve this model from physical point of view. The main attention was paid to various dispersive extensions of shallow water equations, leading to the so-called Boussinesq-type equations, e.g. [11, 12]. However, there are fewer studies which attempt to include the bottom curvature effect into the classical Saint-Venant equations, e.g. [3, 7].

The present study is a further attempt to improve the classical Saint-Venant equations by including a better representation of the bottom shape. As a general derivation procedure, we choose a variational approach based on a relaxed Lagrangian principle [2].

In the next Section, we present the derivation and discussion of some properties of the improved Saint-Venant equations. Then we detail the hyperbolic structure in section 3 and give a numerical example in section 4. Finally, we underline some main conclusions of this study in section 5.

2 Modified Saint-Venant equations

Consider an ideal incompressible fluid of constant density ρ in irrotational motion due to surface gravity waves. The horizontal independent Cartesian variables are denoted by $\boldsymbol{x} = (x_1, x_2)$ and the upward vertical one by y; $y = 0$ corresponds to the still water level. The fluid is bounded below by an impermeable bottom at $y = -d(\boldsymbol{x}, t)$ and above by an impermeable free surface at $y = \eta(\boldsymbol{x}, t)$, the total depth being $h = d + \eta > 0$. The horizontal and vertical velocities are $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ and $v = \phi_y$, where ϕ is a velocity potential and $\boldsymbol{\nabla}$ is the horizontal gradient.

The water wave equations can be derived from a variational principle [9], i.e. as the critical point of a functional $\iiint \mathscr{L}\, \mathrm{d}^2\boldsymbol{x}\, \mathrm{d}t$ involving the relaxed Lagrangian density [2]:

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\eta - \tilde{v})\tilde{\phi} + (d_t + \check{\boldsymbol{u}} \cdot \boldsymbol{\nabla} d + \check{v})\check{\phi} - \frac{g\eta^2}{2} + \int_{-d}^{\eta} \left[\frac{\boldsymbol{u}^2 + v^2}{2} + (\boldsymbol{\nabla} \cdot \boldsymbol{u} + v_y)\phi \right] \mathrm{d}y, \quad (2.1)$$

where g is the acceleration due to gravity force, over 'tildes' and 'wedges' denoting, respectively, quantities traces computed at the free surface $y = \eta$ and at the bottom $y = -d$. We shall also denote below with 'bars' the quantities averaged over the water depth.

In order to simplify the full water wave problem we choose some approximate, but physically relevant, representations of all variables. In this study, we consider very long waves in shallow water. This means that the flow is mainly columnar [10] and that the dispersive effects are negligible. In other words, a vertical slice of the fluid moves somehow like a rigid body. Thus, we choose a simple shallow water ansatz, which is independent of the vertical coordinate y, and such that the vertical velocity v equals the one of the bottom, i.e.,

$$\phi \;\approx\; \bar{\phi}(\boldsymbol{x},t), \qquad \boldsymbol{u} \;\approx\; \bar{\boldsymbol{u}}(\boldsymbol{x},t), \qquad v \;\approx\; \check{v}(\boldsymbol{x},t), \tag{2.2}$$

where $\check{v}(\boldsymbol{x},t)$ is the vertical velocity at the bottom. With this ansatz, the Lagrangian density (2.1) becomes

$$\mathscr{L} \;=\; (h_t + \bar{\boldsymbol{u}}\boldsymbol{\cdot}\boldsymbol{\nabla} h + h\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{u}})\,\bar{\phi} \;-\; \tfrac{1}{2}\,g\,\eta^2 \;+\; \tfrac{1}{2}\,h\,(\bar{\boldsymbol{u}}^2 + \check{v}^2). \tag{2.3}$$

In addition, we impose that the bottom impermeability is fulfilled identically, i.e., we take

$$\check{v} \;=\; -d_t \;-\; \bar{\boldsymbol{u}}\boldsymbol{\cdot}\boldsymbol{\nabla} d. \tag{2.4}$$

Substituting the relation (2.4) into Lagrangian density (2.3), the Euler–Lagrange equations yield

$$\delta\bar{\phi}:\quad 0 \;=\; h_t \;+\; \boldsymbol{\nabla}\boldsymbol{\cdot}[\,h\,\bar{\boldsymbol{u}}\,], \tag{2.5}$$
$$\delta\bar{\boldsymbol{u}}:\quad \boldsymbol{0} \;=\; \bar{\boldsymbol{u}} \;-\; \boldsymbol{\nabla}\bar{\phi} \;-\; \check{v}\,\boldsymbol{\nabla} d, \tag{2.6}$$
$$\delta\eta:\quad 0 \;=\; \bar{\phi}_t \;+\; g\,\eta \;+\; \bar{\boldsymbol{u}}\boldsymbol{\cdot}\boldsymbol{\nabla}\bar{\phi} \;-\; \tfrac{1}{2}\,(\bar{\boldsymbol{u}}^2 + \check{v}^2). \tag{2.7}$$

Taking the gradient of (2.7) and eliminating $\bar{\phi}$ from (2.6) yields the system of governing equations

$$h_t \;+\; \boldsymbol{\nabla}\boldsymbol{\cdot}[\,h\,\bar{\boldsymbol{u}}\,] \;=\; 0, \tag{2.8}$$
$$\partial_t\,[\,\bar{\boldsymbol{u}} - \check{v}\,\boldsymbol{\nabla} d\,] \;+\; \boldsymbol{\nabla}\,[\,g\,\eta \;+\; \tfrac{1}{2}\,\bar{\boldsymbol{u}}^2 \;+\; \tfrac{1}{2}\,\check{v}^2 \;+\; \check{v}\,d_t\,] \;=\; 0, \tag{2.9}$$

together with the auxiliary relations

$$\bar{\boldsymbol{u}} \;=\; \boldsymbol{\nabla}\bar{\phi} \;+\; \check{v}\,\boldsymbol{\nabla} d, \qquad \check{v} \;=\; -\,d_t \;-\; \bar{\boldsymbol{u}}\boldsymbol{\cdot}\boldsymbol{\nabla} d \;=\; -\left[\,d_t \;+\; (\boldsymbol{\nabla}\bar{\phi})\boldsymbol{\cdot}(\boldsymbol{\nabla} d)\,\right]\left[\,1 \;+\; |\boldsymbol{\nabla} d|^2\,\right]^{-1}.$$

Further details on these equations and their variants can be found in [4].

3 Hyperbolic structure

From now on, we consider the equations (2.8) and (2.9) posed in 1D space for simplicity:

$$\partial_t\,h \;+\; \partial_x\,[\,h\,\bar{u}\,] \;=0, \tag{3.1}$$
$$\partial_t\,[\,\bar{u} \;-\; \check{v}\,\partial_x\,d\,] \;+\; \partial_x\left[\,g\,\eta \;+\; \tfrac{1}{2}\,\bar{u}^2 \;+\; \tfrac{1}{2}\,\check{v}^2 \;+\; \check{v}\,\partial_t\,d\,\right] \;=0. \tag{3.2}$$

In order to make appear conservative variables, we introduce the potential velocity variable $U = \bar{\phi}_x$. From equation (2.6) it is straightforward to see that U satisfies the relation $U = \bar{u} - \check{v} d_x$. Depth averaged and vertical bottom velocities can be also easily expressed in terms of U. Consequently, using this new variable equations (3.1), (3.2) can be rewritten as a system of conservation laws

$$\partial_t\,w \;+\; \partial_x\,f(w) \;=\; 0, \tag{3.3}$$

where we introduced the vector of conservative variables w and the advective flux $f(w)$

$$w \;=\begin{pmatrix} h \\ U \end{pmatrix}, \qquad f(w) = \begin{pmatrix} h\,[U - d_t d_x]\left[1 + d_x^2\right]^{-1} \\ g(h-d) + \tfrac{1}{2}\left[U^2 - 2U d_x d_t - d_t^2\right]\left[1 + d_x^2\right]^{-1} \end{pmatrix}.$$

The Jacobian matrix of the advective flux $f(w)$ can be easily computed:

$$\mathbb{A}(w) \;\equiv\; \frac{\partial\,f(w)}{\partial w} \;=\; \frac{1}{1 + d_x^2}\begin{bmatrix} U - d_t d_x & h \\ g\,(1 + d_x^2) & U - d_t d_x \end{bmatrix}.$$

The matrix $\mathbb{A}(w)$ has two distinct eigenvalues:

$$\lambda^{\pm} \;=\; [\,U \,-\, d_t\, d_x\,]\,\bigl[\,1 \,+\, d_x^2\,\bigr]^{-1} \;\pm\; c \;=\; \bar{u} \;\pm\; c, \qquad c^2 \;\equiv\; g\,h\,\bigl[\,1 \,+\, d_x^2\,\bigr]^{-1}.$$

Physically, the quantity c represents the phase celerity of long gravity waves. In the framework of the Saint-Venant equations, it is well known that $c = \sqrt{gh}$. Both expressions differ by the factor $\bigl[1 + d_x^2\bigr]^{-1/2}$. Thus, in the modified Saint-Venant model, the long waves are slowed down by bathymetric variations since fluid particles are constrained to follow the seabed.

4 Numerical experiment

The equations are solved numerically with a spacial finite volume scheme together with a high-order adaptive time stepping. A higher-order spatial scheme is obtained using a piecewise polynomial representation. This is achieved by various so-called reconstruction procedures [6, 8, 14]. In order to solve numerically the last system of equations, we apply a third-order Runge–Kutta scheme with four stages, with an embedded second-order method which is used to estimate the local error and, thus, to adapt the time step size. The model details and its performance can be found in [5]. Several tests and comparisons will be presented at the conference.

Here, we focus of an illustrative example of the new modified Saint-Venant model: Wave generation by a sudden bottom uplift. This simple situation has some important implications to tsunami genesis problems. The bottom is given by the function

$$d(x,t) \;=\; d_0 \;-\; a\,T(t)\;\mathrm{H}(b^2 - x^2)\,\Bigl[\,(x/b)^2 \,-\, 1\,\Bigr]^2, \qquad T(t) \;=\; 1 \;-\; \mathrm{e}^{-\alpha t},$$

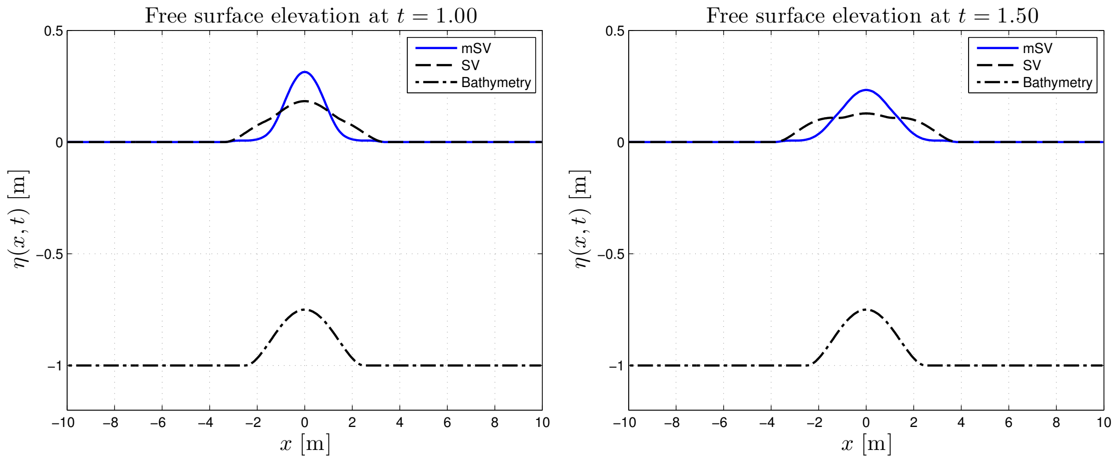
where $\mathrm{H}(x)$ is the Heaviside step function, a is the deformation amplitude and b is the half-length of the uplifting sea floor area. The function $T(t)$ provides a complete information on the dynamics of the bottom motion.

Initially the free surface is undisturbed and the velocity field is zero. Some numerical results of the moving bottom test-case are shown on Figure 1. In this case the differences between the two models are obvious. The modified Saint-Venant equations give a wave with almost twice higher amplitude. This is due to the fact that the wave propagates slower in the region of strong bathymetry gradients in the mSV model. Some differences in the wave shape persist even during the propagation. This test-case clearly shows the differences between the classical and modified Saint-Venant equations. Many other differences and their physical implications will be discussed at the workshop.

5 Conclusion

We derived a modified model of shallow water type which takes into account significant bathymetric variations. Previously, some attempt was already made in the literature to derive shallow water systems for arbitrary slopes and curvature [3, 7]. However, our study contains a certain number of new elements with respect to the existing state of the art. First, our derivation procedure relies on a generalised Lagrangian principle of the water wave problem [2]. Second, we do not introduce any small parameter and our approximation is made through the choice of a suitable constrained ansatz. Third, the resulting governing equations have a simple form and physically sound structure. Fourth, new effects (e.g., speed decrease due to bottom slope) are predicted and should be validated by further investigations.

The proposed model is discretised with a finite volume scheme with adaptive time stepping to capture the underlying complex dynamics. The performance of this scheme will be illustrated on several test cases [5] at the conference, as well as some implications to tsunami modelling.



(a) $t = 1.0\,\mathsf{s}$

(b) $t = 1.5\,\mathsf{s}$

Figure 1: Bottom uplift test-case ($\alpha_2 = 12\,\mathsf{s}^{-1}$).

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