On the interfacial viscous ship waves pattern

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Highlights:

- With the method of stationary phase or steepest descents, the two different asymptotic representations of the interfacial viscous ship wave elevation are derived for the regions inside and outside the Kelvin wedge. Near the cusp lines, the wave elevation at a greater distance can be expressed in terms of Airy integral.
- Using the method developed by Chester, Friedman & Ursell, an uniform asymptotic expansion which spans three separate domains away from and near the cusp lines is obtained. It is convenient to be used for description in the far-field ship waves.

1 Introduction

To study the combined effects of an upper fluid and viscosity on the waves generated by a submerged body, the two semi-infinite fluids system of different densities were used to derive the integral solutions of interfacial elevation due to a point force moving in the lower viscous fluid by Lu and Chwang(2005, 2007). The azimuth angle is denoted as θ and $\theta = 0$ is the moving path of the point force. In the region with $|\theta| < \theta_c = \arctan\sqrt{1/8}$, two real saddle points exist for phase function which appears in the denominator of the integral solutions. When $|\theta| > \theta_c$, two conjugate complex saddle points exist. When $|\theta| \to \theta_c$, two saddle points coincide. The method of stationary phase or steepest descents is used to derive the asymptotic representation of the far-filed waves profiles when θ is away from θ_c and Airy integral method is employed when $|\theta| \approx \theta_c$, but these two kinds of asymptotic expressions are discontinuous across $\pm\theta_c$.

The ordinary method of steepest descents is extended to treat the case of two coalescing saddle points by Chester, Friedman & Ursell(1957), which is referred as the CFU method in the following. Based on our experience of evaluating the capillary gravity time domain Green function(Dai & Chen, 2013), we apply CFU method to obtain the uniform asymptotic expansion of interfacial viscous ship waves at a large horizontal distance.

2 Mathematical expressions of interfacial viscous ship waves

The two-fluid system with an upper inviscid fluid and a lower viscous fluid is assumed to be incompressible, homogenous and stable. Cartesian coordinates are taken on the undisturbed interface between the upper and lower fluid, the x axis points to the direction of the moving point force and the z axis vertically upward. The point force $\mathrm{F} = (-F, 0, 0)$ is located at $(0, 0, -z_0)$ with $z_0 > 0$ and moving at a constant velocity U. The density of the upper and lower fluid is denote as ρ_1 and ρ_2 respectively, $\sigma = \rho_1/\rho_2$. The dynamic viscosity of the lower fluid is given as μ , $\varepsilon = \mu g/(\rho_2 U^2)$ with g the gravitational acceleration .

The Laplace equation is taken as the governing equation for inviscid flows while the steady Oseen equations are taken for viscous flows. By applying the Fourier integral transform, the elevation of interfacial ship waves η can be expressed as a double integral(Lu & Chwang , 2007)

$$\eta = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{A}{D}\exp(\mathrm{i}Rf)\mathrm{d}\alpha\mathrm{d}\beta \tag{1}$$

where

$$A = F\lambda[(\mathrm{i}\alpha + 2\varepsilon K^2)\exp(-Kz_0) - 2\varepsilon KB\exp(-Bz_0)], \tag{2}$$
$$D = \gamma K - \alpha^2 + 4\mathrm{i}\varepsilon\alpha K^2\lambda + 4\varepsilon^2 K^3(K - B)\lambda, \tag{3}$$
$$f(\alpha, \beta) = \alpha\cos\theta + \beta\sin\theta, \tag{4}$$
$$K = \sqrt{\alpha^2 + \beta^2}, \tag{5}$$
$$B = \sqrt{\mathrm{i}\alpha/\varepsilon + K^2}, \tag{6}$$
$$\lambda = 1/(1 + \sigma), \tag{7}$$
$$\gamma = (1 - \sigma)/(1 + \sigma). \tag{8}$$

(R, θ) are the cylindrical coordinates on the horizontal (x,y) plane such that $x = R\cos\theta$ and $\mathrm{y} = \mathrm{R}\sin\theta$.

3 Nonuniform asymptotic expansion of ship waves

For small ε, the dispersion function D has two zeros with respect to α,

$$\alpha_j(\beta) = (-1)^{j+1} a_0(\beta) + \mathrm{i}\varepsilon a_1(\beta) + O(\varepsilon^{3/2}), (\mathrm{j} = 1, 2) \tag{9}$$

where

$$a_0(\beta) = \sqrt{(\gamma^2 + \gamma\sqrt{\gamma^2 + 4\beta^2})/2}, \tag{10}$$

$$a_1(\beta) = \frac{4\lambda a_0^6(\beta)}{2a_0^2(\beta) - \gamma^2}. \tag{11}$$

Using the Cauchy residue theorem, the leading terms which contribute significantly to the far-field wave profiles can be written as

$$\eta = \frac{F\lambda}{2\pi} \sum_{j=1}^{2} \int_{-\infty}^{+\infty} \mathrm{d}\beta \frac{a_0^2(\beta) N_j(\beta)}{2a_0^2(\beta) - \gamma^2} \exp[-\varepsilon a_1(\beta) R\cos\theta + \mathrm{i}R f_j], \tag{12}$$

where

$$\begin{aligned} N_j(\beta) = {} & [1 + 2(-1)^j \mathrm{i}\varepsilon\gamma^{-2} a_0^3(\beta)] \exp[-\gamma^{-1} a_0^2(\beta) z_0] - 2(-1)^j \mathrm{i}\sqrt{\varepsilon}\gamma^{-1} a_0^{3/2}(\beta) \\ & \times \exp\left\{-\sqrt{a_0(\beta)/2\varepsilon}[1 - (-1)^j \mathrm{i}] z_0 - (-1)^j \pi\mathrm{i}/4\right\}, \end{aligned} \tag{13}$$

$$f_j(\beta, \theta) = (-1)^{j+1} a_0(\beta)\cos\theta + \beta\sin\theta. \tag{14}$$

If $|\theta| < \theta_c$, there are two real saddle points for phase function $f_j(\beta,\theta)$ (j=1,2), which are given by

$$\beta_{jk} = (-1)^j \gamma q_k \sqrt{(q_k + 1)/2}\tan\theta, (k = 1, 2), \tag{15}$$

where

$$q_k = 2\Big[1 + (-1)^{j+k+1}\sqrt{1 - 8\tan^2\theta}\Big]^{-1}. \tag{16}$$

If $\theta_c < |\theta| < \pi/2$, two complex saddle points are conjugate

$$\beta_{jk} = (-1)^j \gamma q_k \sqrt{q_k + 1}\tan\theta/\sqrt{2}, (k = 1, 2), \tag{17}$$

where

$$q_k = 2\Big[1 + (-1)^{j+k+1}\mathrm{i}\sqrt{8\tan^2\theta - 1}\Big]^{-1}. \tag{18}$$

If $|\theta| = \theta_c$, these points coalesce into a single saddle point of order 2

$$\beta_{jc} = (-1)^j \mathrm{sgn}(\theta)\sqrt{3}\gamma/2. \tag{19}$$

When $|\theta| < \theta_c$, we have the asymptotic solution for waves within the Kelvin wedge

$$\begin{aligned} \eta = {} & \frac{F\lambda}{\sqrt{2\pi R}} \sum_{j=1}^{2}\sum_{k=1}^{2} \frac{a_0^2(\beta_{jk}) N_j(\beta_{jk})}{[2a_0^2(\beta_{jk}) - \gamma^2]\sqrt{\left|f_j^{(2)}(\beta_{jk},\theta)\right|}} \\ & \times \exp\Big[-\varepsilon a_1(\beta_{jk}) R\cos\theta + \mathrm{i}R f_j(\beta_{jk},\theta) + \mathrm{i}\cdot\mathrm{sgn}(f_j^{(2)}(\beta_{jk},\theta))\frac{\pi}{4}\Big]. \end{aligned} \tag{20}$$

When $|\theta| \approx \theta_c$, We have the asymptotic solution for waves near the cusp lines ($|\theta| = \theta_c$)

$$\begin{aligned} \eta = {} & F\lambda \sum_{j=1}^{2} \frac{a_0^2(\beta_{jc}) N_j(\beta_{jc})}{2a_0^2(\beta_{jc}) - \gamma^2} \left(\frac{2}{R\left|f_j^{(3)}(\beta_{jc},\theta)\right|}\right)^{1/3} \\ & \times \mathrm{Ai}\left[f_j^{(1)}(\beta_{jc},\theta)\left(\frac{2R^2}{f_j^{(3)}(\beta_{jc},\theta)}\right)^{1/3}\right] \exp\left[-\varepsilon a_1(\beta_{jc}) R\cos\theta + \mathrm{i}R f_j(\beta_{jc},\theta)\right], \end{aligned} \tag{21}$$

which Ai ($\bullet$) is the Airy function defined in Abramowita & Stegun (1967). When $\theta_c < |\theta| < \pi/2$, We have the asymptotic solution for waves outside the Kelvin wedge

$$\eta = \frac{F\lambda}{\sqrt{2\pi R}} \sum_{j=1}^{2} \frac{a_0^2(\beta_{j2}) N_j(\beta_{j2})}{[2a_0^2(\beta_{j2}) - \gamma^2] \sqrt{\left|f_j^{(2)}(\beta_{j2},\theta)\right|}} \exp\left[-\varepsilon a_1(\beta_{j2}) R\cos\theta + \mathrm{i} R f_j(\beta_{j2},\theta)\right]$$
$$\times \exp\left\{\frac{\mathrm{i}}{2}\mathrm{sgn}\left[\mathrm{Im}(\mathrm{i} f_j^{(2)}(\beta_{j2},\theta)\right]\pi\right\} \exp\left\{-\frac{\mathrm{i}}{2}\arg\left[\mathrm{i} f_j^{(2)}(\beta_{j2},\theta)\right]\right\}. \tag{22}$$

As θ tends to $\pm\theta_c$, the order of R changes from $1/2$ in (20) and (22) to $1/3$ in (21) and the behavior of these approximations differs radically, a uniform approximation valid for $||\theta| - \theta_c| < V$ with V a finite value is desired.

4 Uniform asymptotic expansions of ship waves

We apply the CFU method to develop the uniform asymptotic of interfacial waves η in the far field. A cubic transform of the variable of integration β to t is defined

$$\mathrm{i} f_j(\beta,\theta) = -\Big(\frac{t^3}{3} - u_j^2 t\Big) + \rho_j. \tag{23}$$

It follows that

$$\eta = \sum_{j=1}^{2} \mathrm{e}^{R\rho_j} \int_{\infty \mathrm{e}^{2\pi\mathrm{i}/3}}^{\infty \mathrm{e}^{4\pi\mathrm{i}/3}} G_{0j}(t,\theta) \exp[-R(\frac{t^3}{3} - u_j^2 t)]\mathrm{d}t \tag{24}$$

where

$$G_{0j}(t,\theta) = \frac{F\lambda}{2\pi} \frac{a_0^2(\beta) N_j(\beta)}{2a_0^2(\beta) - \gamma^2} \exp[-\varepsilon a_1(\beta) R\cos\theta] \frac{\mathrm{d}\beta}{\mathrm{d}t} \tag{25}$$

A Bleistein sequence is established to replace the integrand $G_{0j}(t,\theta)$

$$G_{0j}(t,\theta) = b_{0j} + b_{1j} t + (t^2 - u_j^2) H_{0j}(t,\theta). \tag{26}$$

We obtain the uniform asymptotic expansions of η for large R

$$\eta \approx \sum_{j=1}^{2} 2\pi\mathrm{i}\mathrm{e}^{R\rho_j} \Big[\frac{b_{0j}}{R^{1/3}} \mathrm{Ai}(R^{2/3} u_j^2) + \frac{b_{1j}}{R^{2/3}} \mathrm{Ai}'(R^{2/3} u_j^2)\Big], \tag{27}$$

where

$$\rho_j = \mathrm{i}(f_j(\beta_{j1},\theta) + f_j(\beta_{j2},\theta)/2, \tag{28}$$

$$b_{0j} = \begin{cases} \frac{1}{2} Q_j\Big|_{\beta=\beta_{j2}} \frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=u_j} + \frac{1}{2} Q_j\Big|_{\beta=\beta_{j1}} \frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=-u_j}, & (\theta \neq \theta_\mathrm{c}), \\ Q_j\Big|_{\beta=\beta_{jc}} \frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=0}, & (\theta = \pm\theta_\mathrm{c}), \end{cases} \tag{29}$$

$$b_{1j} = \begin{cases} \frac{1}{2u_j} Q_j\Big|_{\beta=\beta_{j2}} \frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=u_j} - \frac{1}{2u_j} Q_j\Big|_{\beta=\beta_{j1}} \frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=-u_j}, & (\theta \neq \theta_\mathrm{c}), \\ \frac{\mathrm{d}Q_j}{\mathrm{d}\beta}\Big|_{\beta=\beta_{jc}} \Big(\frac{\mathrm{d}\beta}{\mathrm{d}t}\Big)^2\Big|_{t=0} + Q_j\Big|_{\beta=\beta_{jc}} \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2}\Big|_{t=0}, & (\theta = \pm\theta_\mathrm{c}), \end{cases} \tag{30}$$

$$u_i = \begin{cases} \exp(\pi\mathrm{i}/2)[(f_j(\beta_{j1},\theta) - f_j(\beta_{j2},\theta))3/4]^{1/3}, & (|\theta| < \theta_\mathrm{c}), \\ 0, & (|\theta| = \theta_\mathrm{c}), \\ \exp(\pi\mathrm{i})[\mathrm{Im}(f_j(\beta_{j1},\theta) - f_j(\beta_{j2},\theta))3/4]^{1/3}, & (\pi/2 > |\theta| > \theta_\mathrm{c}). \end{cases} \tag{31}$$

with

$$Q_j = \frac{F\lambda}{2\pi} \frac{a_0^2(\beta) N_j(\beta)}{2a_0^2(\beta) - \gamma^2} \exp(-\varepsilon a_1(\beta) R\cos\theta), \tag{32}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=\mp u_j} = \begin{cases} \exp(-\pi\mathrm{i}/2)\Big|2u_j \Big/ f_j^{(2)}(\beta_{j1,2},\theta)\Big|^{1/2}, & (|\theta| < \theta_\mathrm{c}) \\ \exp[-(\pi \pm \theta_T)\mathrm{i}/2]\Big|2u_j \Big/ f_j^{(2)}(\beta_{j1,2},\theta)\Big|^{1/2}, & (\theta_c < |\theta| < \pi/2). \end{cases} \tag{33}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}\Big|_{t=0} = \exp(3\pi\mathrm{i}/2)\left[2/f_j^{(3)}(\beta_{jc},\pm\theta_\mathrm{c})\right]^{1/3}, \qquad (\theta = \pm\theta_\mathrm{c}), \tag{34}$$

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2}\Big|_{t=0} = f_j^{(4)}(\beta_{jc},\pm\theta_\mathrm{c}) \Big/ \left[54^{1/5} f_j^{(3)}(\beta_{jc},\pm\theta_c)\right]^{3/5}, \qquad (\theta = \pm\theta_\mathrm{c}). \tag{35}$$

In (32), the phase θ_T is defined as $\arg[i/f_j^{(2)}(\beta_{j2},\theta)]$ and contained in $[-\pi,\pi]$.

5 Discussions

The interfacial viscous waves present the same behavior as the Kelvin wave as described by Ursell (1960). The cusp lines separate the oscillatory wave motion inside Kelvin wedge behind from an exponentially small motions outside the Kelvin wedge, and the transition region near the cusp line can be described by the Airy integral and corresponds to a coalescent pair of saddle points in the Fourier integral. By the use of CFU method, a uniform asymptotic expansion spanning the whole region is obtained. Using appropriate expansions for the Airy function , (27) is consistent with (20) for $|\theta| > \theta_c$ and (22) for $|\theta| < \theta_c$.

It is found the transverse waves are obvious in the Kelvin wedge and the largest waves are near the cusp lines in Figs.1-4. As the ratio of density increases ,the wave amplitude becomes small as indicated in Figs. 1-2. As the viscosity of the lower fluid decreases, the wave amplitude becomes large and much more diverge waves are found near the cusp lines as illustrated in Figs. 3-4.

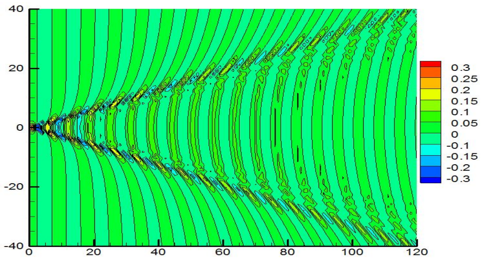

Figure 1: Contours of the wave elevation with $\sigma = 0.01, \varepsilon = 1.0e-4, \mathrm{z}_0 = 0.5$.

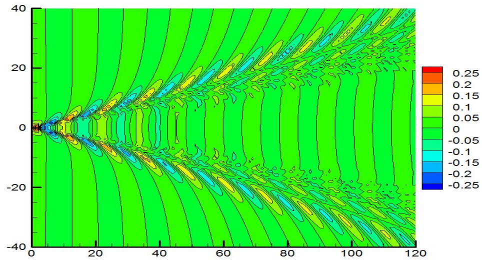

Figure 2: Contours of the wave elevation with $\sigma = 0.3, \varepsilon = 1.0e-4, \mathrm{z}_0 = 0.5$.

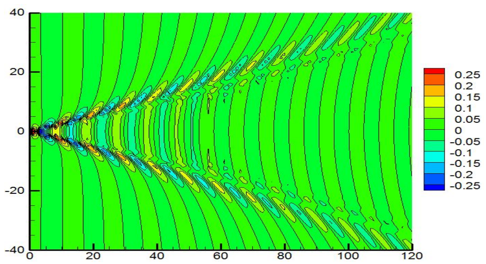

Figure 3: Contours of the wave elevation with $\sigma = 0.2, \varepsilon = 3.0e-3, \mathrm{z}_0 = 0.4$.

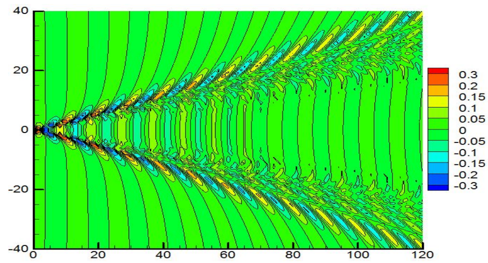

Figure 4: Contours of the wave elevation with $\sigma = 0.2, \varepsilon = 1.0e-7, \mathrm{z}_0 = 0.4$.

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