A semi-analytic formulation for the hydrodynamic diffraction by submerged ellipsoids

by

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"The ellipsoid is god's gift to naval architects"
Georg Weinblum (1897-1974); quoted by JN Newman (1972)

1 Introduction

In this study we present some initial results of an ongoing effort towards an analytical approach for solving both the wave diffraction and the wave resistance problems by submerged tri-axial ellipsoids moving rectilinearly in water of infinite depth below a free upper surface. The effort undertaken is by no means trivial, as there are yet several concealed aspects related to ellipsoidal harmonics represented by the Lamé functions. Many mathematical properties of the Lamé functions are still unknown or ill-understood which makes their use a very difficult task. Only recently, 80 years after Hobson's pioneering book [1], a new state of the art contribution on Ellipsoidal Harmonics by the second author [2] was presented. In fact, the content of this book allowed the elaboration of the titled subject. The existing contributions on the hydrodynamics of ellipsoidal vessels using analytical formulations are rather limited. Among the very few, we mention the works due to Havelock [3] on the wave resistance problem and that of the last author [4], which investigated maneuvering aspects of ellipsoidal forms. Yet, these contributions stopped at the formulation stage without implementing numerically the outlined theory. Moreover, the effect of proximity to the free-surface is not fully considered. It is the authors' intention to cover part of the existing gap on the subject and the paper at hand is a first step in this direction. Here we chose to investigate only the wave diffraction problem, although the formulation we use is generic and can also be employed for the wave resistance problem, as well as for the general problem of a steadily moving tri-axial ellipsoid under waves. The same formulation can in principle be extended to tackle the case of water of finite (constant) depth. The only requirement in that respect is to use the proper kernel in the expression for the Green's function. The efficiency of the following described methodology relies on the employment of ultimate image singularity method that was developed by the last author [5] for arbitrary external ellipsoidal harmonics.

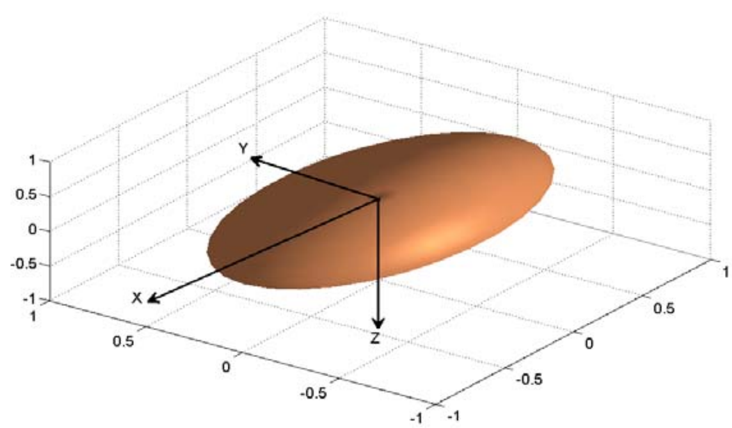

Figure 1 3D image of an ellipsoid with a_1=1, a_2=0.55, a_3=0.2

2 Multipole expansions in curvilinear coordinates for surface waves

The ellipsoid (Fig. 1) is considered immersed at a distance f below the undisturbed free surface of a liquid field with infinite depth and subjected to the action of monochromatic incident waves with linear amplitude A, frequency ω and arbitrary heading angle β with respect to the ellipsoid's major axis. We use a left-handed Cartesian (x, y, z) coordinate system, fixed on the free surface with x parallel to the undisturbed free surface and z pointing in the gravity direction. We start with the Green's function for the concerned problem

$$G(x,y,z)=\frac{1}{\sqrt{x^2+y^2+(z-f)^2}}-\frac{1}{2\pi}\int_0^{\infty}\int_{-\pi}^{\pi}\frac{K+k}{K-k}e^{-k(z+f)+ik(x\cos a+y\sin a)}\mathrm{d}a\mathrm{d}k\,, \tag{1}$$

that satisfies the linearized free surface and the rigid bottom boundary conditions. The far-field radiation condition of outgoing waves at infinity is satisfied if we circumvent the simple pole in (1) such that the Cauchy integral is expanded as

$$\int_0^{\infty}\frac{F(k)}{k-\sigma}\mathrm{d}k = PV\int_0^{\infty}\frac{F(k)}{k-\sigma}\mathrm{d}k+i\pi F(\sigma)\,. \tag{2}$$

In (1), $K=\omega^2/g$ and g is the gravitational acceleration whilst in (2) PV denotes the Cauchy Principal Value Integral. Our goal is to find the eigen-expansion terms G_n^m of the Green's function that will accordingly allow for the formulation of the diffraction component. To this end we employ the theorem shown in [5] according to which

$$F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)=-\int_{s_0}\frac{s_n^m(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\,, \tag{3}$$

$$F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)=\frac{\partial}{\partial z}\int_{s_0}\frac{d_n^m(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\,, \tag{4}$$

where E_n^m and F_n^m denote the Lamé functions of the first and the second kind respectively of degree n and order m. Equations (3) and (4) correspond to even (classes **K** and **L**) and odd (classes **M** and **N**) of Lamé functions, whilst ρ, μ, ν are the orthogonal ellipsoidal coordinates. For the definition of s_n^m (sources) and d_n^m (doublets) we refer the reader to [5]. The integration area s_o is the area of the fundamental ellipse

$$\frac{\xi^2}{h_2^2}+\frac{\eta^2}{h_2^2-h_3^2}=1\,, \tag{5}$$

where h_2 and h_3 depend to the three axes and are given in [5].

The use of the theorem shown by the last author in [5] yields the eigen-expansions of the Green's function as

$$G_n^m=F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)+\frac{1}{2\pi}\int_{s_0}g_n^m(\xi,\eta)\int_{-\pi}^{\pi}\int_0^{\infty}\hat{Q}(k)e^{-\Omega}\mathrm{d}k\mathrm{d}a\mathrm{d}\xi\mathrm{d}\eta\,, \tag{6}$$

where g_n^m denotes alternatively s_n^m and d_n^m. Also $\hat{Q}(k)=(K+k)/(K-k)$ and $\hat{Q}(k)=k(K+k)/(K-k)$ are used for the even and odd harmonics respectively whereas $\Omega=k(z+2f)-ik\left[(x-\xi)\cos a+(y-\eta)\sin a\right]$.

3 Green's function expansion in ellipsoidal harmonics

To have a full expansion in terms of ellipsoidal harmonics, the regular terms of (6) must be processed accordingly. To this end we let

$$e^{-kz+ik(x\cos a+y\sin a)}=\sum_{s=0}^{\infty}\sum_{t=1}^{2s+1}A_s^t(k,a)\mathbb{S}_s^t(\mu,\nu)E_s^t(\rho);\quad \mathbb{S}_s^t(\mu,\nu)=E_s^t(\mu)E_s^t(\nu)\,, \tag{7}$$

where $\mathbb{S}_s^t(\mu,\nu)$ denotes the surface Lamé functions. The ill-understood properties of these functions allow only numerical elaboration of (7) to calculate the expansion coefficients $A_s^t(k,a)$. These are obtained by employing (7) exactly on the surface of the ellipsoid, namely letting $\rho=a_1$ with a_1 being the longitudinal (major) semi-axis of the solid. However, the employment of the orthogonality property of the Lamé functions is needed which is [2]

$$\int_{S_\rho} \mathbb{S}_n^m(\mu,\nu)\mathbb{S}_s^t(\mu,\nu)\mathrm{d}\Omega(\mu,\nu) = \gamma_n^m \delta_{ns}\delta_{mt}\,, \tag{8}$$

$$\mathrm{d}\Omega(\mu,\nu) = \frac{(\mu^2-\nu^2)\mathrm{d}\mu\mathrm{d}\nu}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}\,, \tag{9}$$

where δ denotes Kroneker's delta function and γ are the orthogonality constants. Again, the only advisable option is to compute the orthogonality constants numerically and the surface integral over the surface of the ellipsoid must be split to the eight octants. With the above remarks the eigen-expansion of the Green's function becomes

$$G_n^m = F_n^m(\rho)\mathbb{S}_n^m(\mu,\nu) + \sum_{s=0}^{\infty}\sum_{t=1}^{2s+1} C_{ns}^{mt}\mathbb{S}_s^t(\mu,\nu)E_s^t(\rho)\,, \text{ where} \tag{10}$$

$$C_{ns}^{mt} = \frac{1}{2\pi\gamma_s^t E_s^t(a_1)}\int_{s_0} g_n^m(\xi,\eta)\int_0^\infty \hat{Q}(k)e^{-2kf}\int_{S_\rho} e^{-kz_0}\left[\int_{-\pi}^{\pi} e^{ik\left[(x_0-\xi)\cos a+(y_0-\eta)\sin a\right]}\mathrm{d}a\right]\mathbb{S}_s^t(\mu,\nu)\mathrm{d}\Omega(\mu,\nu)\mathrm{d}k\mathrm{d}\xi\mathrm{d}\eta\,, \tag{11}$$

where $x_0(\mu,\nu)$, $y_0(\mu,\nu)$, $z_0(\mu,\nu)$ are calculated on the body at $\rho=a_1$. In fact this is not the final analytic form of the expansion coefficients. They must be further manipulated to exhaust analytical expansions including the employment of (2) in order to satisfy the far-filed radiation condition.

4 Velocity potentials

The velocity potential of the incident wave train of frequency ω and linear amplitude A, propagating at angle β with respect to the horizontal-x axis of the ellipsoid, is given by

$$\phi_I = \omega A\frac{1}{K}e^{-Kf}e^{-Kz+iK(x\cos\beta+y\sin\beta)}. \tag{12}$$

The above potential, when expressed with respect the ellipsoidal coordinates using (7), becomes

$$\phi_I = \omega A\frac{1}{K}e^{-Kf}\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} B_n^m(K,\beta)\mathbb{S}_n^m(\mu,\nu)E_n^m(\rho)\,, \tag{13}$$

where

$$B_n^m(K,\beta) = \frac{1}{\gamma_n^m E_n^m(a_1)}\int_{S_\rho} e^{-Kz_0+iK(x_0\cos\beta+y_0\sin\beta)}\mathbb{S}_n^m(\mu,\nu)\mathrm{d}\Omega(\mu,\nu)\,. \tag{14}$$

The Green's function given by (10) allows expressing the diffraction potential as

$$\phi_D = \omega A\frac{1}{K}e^{-Kf}\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} H_n^m\left[\mathbb{S}_n^m(\mu,\nu)F_n^m(\rho) + \sum_{s=0}^{\infty}\sum_{t=1}^{2s+1} C_{ns}^{mt}\mathbb{S}_s^t(\mu,\nu)E_s^t(\rho)\right], \tag{15}$$

where H_n^m are unknown expansion coefficients to be determined from applying the zero velocity kinematical condition on the surface of the ellipsoid. In other words it should hold that

$$\partial\phi_D/\partial\rho = -\partial\phi_I/\partial\rho\,, \tag{16}$$

calculated specifically at $\rho = a_1$. After introducing (12) and (15) into the boundary condition (16) and making use the orthogonality properties of the ellipsoidal harmonics (the Lamé functions of the first kind), (16) yields

$$\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left[\delta_{mt}\delta_{ns} + C_{ns}^{mt}\frac{\dot{E}_s^t(a_1)}{\dot{F}_s^t(a_1)}\right]H_n^m = -B_s^t(K,\beta)\frac{\dot{E}_s^t(a_1)}{\dot{F}_s^t(a_1)}\,, \tag{17}$$

where the upper dot denotes differentiation with respect to the argument. A crucial point that should be mentioned as far the particular solution is concerned, is associated with the fact that the present methodology results in one linear system to be treated for calculating the expansion coefficients, here denoted by H_n^m. It is also reminded that when applying a similar method of image singularities for prolate spheroids using Havelock's theorem [5], the systems that should be solved are two, different for sine and cosine functions.

4 Some initial numerical results

We faced several significant numerical challenges during the implementation of the outlined methodology. Most of them are in connection with the precise computation of Lamé functions for arbitrary degree and order. It is noted that analytic formulae for Lamé functions exist only up to degree seven. Even in this case however, the roots of the characteristic polynomials must be obtained numerically. We have developed an algorithm that properly calculates the Lamé functions of both kinds and all classes, as well as the orthogonality constants. We succeeded to do that for arbitrary large degrees and orders although there are unavoidable limitations due to the large numbers obtained for large degrees, orders and arguments. It should be also remarked that special treatment requires the expansion of arbitrary functions in terms of ellipsoidal harmonics [see (14) and (17)]. This allows only one option to proceed, namely direct numerical integration, which however requires splitting the surface integral to the eight octants and taking into account the change in sign of the Cartesian coordinates. We believe that by this we have highlighted the most important issues of the huge effort undertaken and explained why this work is still ongoing. To the best of our knowledge there have not been any efforts reported in the literature that tried to tackle this particular problem analytically. The only available data for exciting diffraction forces exerted on submerged solids, concern spheroids and spheres and explicitly in infinite water depth. These have been used to validate our method by making the ellipsoid a spheroid using nearly equal semi-minor axes. Relevant results are listed in Table 1 in which the exciting forces are normalized by $\rho g A a^2$. The benchmark data were taken from the work of Wu and Eatock Taylor [6]. It is easily seen that the comparisons are fairly good and this is indeed encouraging. The same level of agreement was observed when comparing our method to simulate the exciting forces exerted on a submerged sphere against those reported in Wang [7]. New results on the exciting forces and moments exerted on "actual" tri-axial ellipsoidal shapes, will be presented during the workshop.

Table 1 Magnitudes of the hydrodynamic exciting forces acting on a fixed prolate spheroid parallel to the free surface; the semi-major to semi-minor axis ratio a/b=6; immersion f=3b; heading angle β=0^{O}; axes dimensions assumed for the ellipsoid a_1=1, a_2=0.1666666, a_3=0.166 and immersion f=3a_3.

	Surge force		Heave force	
Ka_1	Wu & Eatock Taylor [6]	Present	Wu & Eatock Taylor [6]	Present
0.1	0.0116	0.0115	0.0217	0.0220
0.2	0.0220	0.0218	0.0412	0.0419
0.3	0.0313	0.0310	0.0587	0.0598
0.4	0.0394	0.0390	0.0740	0.0757
0.5	0.0464	0.0460	0.0879	0.0897
0.6	0.0526	0.0519	0.0998	0.1019
0.7	0.0577	0.0569	0.1095	0.1122
0.8	0.0614	0.0610	0.1177	0.1207
0.9	0.0654	0.0642	0.1238	0.1275
1.0	0.0680	0.0666	0.1287	0.1326

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