GN equations to describe internal solitary waves in two-layer fluid

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Green-Naghdi (GN) equations in two-layer fluid are derived and used to describe internal solitary waves. Two methods to generate internal waves are studied here. The first method is to set an initial wave profile and velocity given by KdV theory. This method can not be used to generate internal solitary wave of large amplitude. Gravity collapse is used as the second method to generate internal solitary wave of large amplitude. The second method is similar to that in the experiment of Grue *et al.* (1999). The results by GN theory are in good agreement with those by the fully nonlinear model developed in Grue *et al.* (1999).

1 Introduction

In GN theory, nonlinear boundary conditions are satisfied on the instantaneous free surface. Demirbilek and Webster (1992) applied level-2 GN theory to shallow-water 2D problems. Zhao *et al.* (2010) applied the GN level-2 theory to 3D case. Zhao *et al.* (2009) applied the GN level-3 theory to 2D water wave simulations. Furthermore, Zhao and Duan (2012) applied higher level (level-5 and level-7) to 2D shallow-water wave problems and get fully satisfactory results. They achieved that the converged GN results are in excellent agreement with experimental results. Duan and Zhao (2012) applied GN theory to underwater landslide-induced tsunamis.

To authors' knowledge, GN theory has not been applied to flows in two-layer fluid before. For internal solitary waves, Grue *et al.* (1999) made experiments and introduced a fully nonlinear model. Their numerical procedure is started by prescribing a small amplitude and solving the equations iteratively. Then, they solve equations using Newton-Raphson's method with initial guess extrapolated from the previous solution. In this paper, we simulate internal solitary waves by using GN theory in two-layer fluid in time domain.

2 GN Theory

The GN equations for one-layer flow in Webster *et al.* (2011) are given by :

$$\frac{\partial \beta}{\partial t} = \sum_{n=0}^{K} \beta^n \left(w_n - \frac{\partial \beta}{\partial x} u_n \right) \tag{1}$$

$$\frac{\partial}{\partial x}\left(G_n + gS_n\right) + nE_{n-1} - \alpha^n \frac{\partial}{\partial x}\left(G_0 + gS_0\right) + \left(\beta^n - \alpha^n\right)\frac{\partial}{\partial x}\left(\frac{\hat{p}}{\rho}\right) = 0 \quad \text{for} \quad n = 1, 2, \cdots, K \tag{2}$$

where K stands for the level of GN theory, $z = \beta$ for the free surface, $z = \alpha$ for the bottom, expressions of E_n, S_n and G_n can be found in Webster *et al.* (2011).

For two-layer problems, we set the bottom as $z = \alpha(x, t)$, the interface as $z = \beta(x, t)$ and the free surface as $z = \gamma(x, t)$. The GN equations in two-layer fluid are then for upper layer :

$$\frac{\partial \gamma}{\partial t} = \sum_{n=0}^{K^U} \gamma^n \left(w_n^U - \frac{\partial \gamma}{\partial x} u_n^U \right) \tag{3}$$

$$\frac{\partial}{\partial x}\left(G_n^U + gS_n^U\right) + nE_{n-1}^U - \beta^n \frac{\partial}{\partial x}\left(G_0^U + gS_0^U\right) + \left(\gamma^n - \beta^n\right)\frac{\partial}{\partial x}\left(\frac{\hat{p}^U}{\rho^U}\right) = 0 \tag{4}$$

for $n = 1, 2, \cdots, K^U$ with K^U as the GN level in upper layer, and for lower layer :

$$\frac{\partial \beta}{\partial t} = \sum_{n=0}^{K^L} \beta^n \left(w_n^L - \frac{\partial \beta}{\partial x} u_n^L \right) \tag{5}$$

$$\frac{\partial}{\partial x}\left(G_n^L + gS_n^L\right) + nE_{n-1}^L - \alpha^n \frac{\partial}{\partial x}\left(G_0^L + gS_0^L\right) + \left(\beta^n - \alpha^n\right)\frac{\partial}{\partial x}\left(\frac{\hat{p}^L}{\rho^L}\right) = 0 \tag{6}$$

for $n = 1, 2, \cdots, K^L$ with K^L as the GN level in the lower layer.

The dynamic boundary conditions are

$$\hat{p}^L = p|_{z=\beta} = \bar{p}^U = \rho^U G_0^U + \rho^U g S_0^U + \hat{p}^U \tag{7}$$

$$\hat{p}^U = p|_{z=\gamma} = 0 \tag{8}$$

ensuring the continuity of pressure through the interface and pressure at the free surface.

3 Numerical algorithm

The equations (4) and (6) make a system of $(K^U + K^L)$ coupled, quasi-linear partial differential equations for the $(K^U + K^L)$ unknowns. The unknowns are expressed here as a $(K^U + K^L)$-dimensioned vector, $\boldsymbol{\xi}(x, t)$ and the equations are rewritten in the form as :

$$\tilde{\mathbf{A}}\dot{\boldsymbol{\xi}}_{,xx} + \tilde{\mathbf{B}}\dot{\boldsymbol{\xi}}_{,x} + \tilde{\mathbf{C}}\dot{\boldsymbol{\xi}} = \tilde{\mathbf{f}} \tag{9}$$

where,

$$\dot{\boldsymbol{\xi}} = [\dot{u}_0^U, \quad \dot{u}_1^U, \quad \cdots, \quad \dot{u}_{K^U-1}^U, \quad \dot{u}_0^L, \quad \dot{u}_1^L, \quad \cdots, \quad \dot{u}_{K^L-1}^L]^T \tag{10a}$$

$$\dot{\boldsymbol{\xi}}_{,x} = [\dot{u}_{0,x}^U, \quad \dot{u}_{1,x}^U, \quad \cdots, \quad \dot{u}_{K^U-1,x}^U, \quad \dot{u}_{0,x}^L, \quad \dot{u}_{1,x}^L, \quad \cdots, \quad \dot{u}_{K^L-1,x}^L]^T \tag{10b}$$

$$\dot{\boldsymbol{\xi}}_{,xx} = [\dot{u}_{0,xx}^U, \dot{u}_{1,xx}^U, \cdots, \dot{u}_{K^U-1,xx}^U, \dot{u}_{0,xx}^L, \dot{u}_{1,xx}^L, \cdots, \dot{u}_{K^L-1,xx}^L]^T \tag{10c}$$

and $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{f}}$ are functions of x and $\boldsymbol{\xi}$ and its spatial derivatives which are not given here for the sake of simplicity. This system can be easily solved by Thomas algorithm used in Demirbilek & Webster (1992). This algorithm is further improved in Zhao & Duan (2012).

4 Test cases

In this section, we will give some numerical simulations of internal solitary waves. The first method we used here is to set an initial wave shape on the interface. The initial shape and velocity are from the KdV Theory given in Long (1956) and in Grue *et al.* (1999).

We want to reproduce the same solitary wave generated by Grue *et al.* (1999). Their physical experiment are calibrated with a layer of fresh water above a layer of brine. The depth of the brine is $h_1 = 62$cm and the depth of the fresh water is $h_2 = 15$cm. The density of brine is 1.022g/cm^3 and the density of fresh water is 0.999g/cm^3.

On Figure 7 in their paper, the wave profile is given at a fixed position. Five wave profiles with $a/h_2 =$ 0.22, 0.36, 0.91, 1.23 and 1.51 with a as the internal solitary amplitude, are shown. The cases with $a/h_2 =$ 0.22 and $a/h_2 = 0.91$ are simulated here by using GN theory.

For $a/h_2 = 0.22$, the initial shape with amplitude $a = 0.22h_2 = 0.33$m is shown on the left of Figure 1. After 600 seconds propagation, an internal solitary wave of amplitude = 0.0351m is shown in the middle of Figure 1 (Note that the initial solitary wave amplitude is 0.033m). The KdV results with amplitude = 0.0351m is shown on the right of Figure 1.

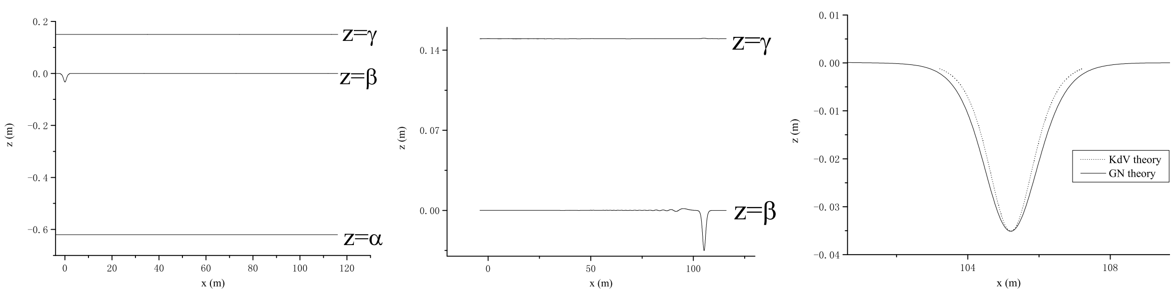

Figure 1: Initial wave profile with $a = 0.22h_2$ from KdV theory (left), snapshot at $t = 600$s (middle) and comparison with KdV results (right)

For $a/h_2 = 0.91$, we use the same method to simulate internal waves. The amplitude of initial wave is $a = 0.91h_2 = 0.1365$m at $t = 0$s. At $t = 550$s, the snapshot of wave train is shown on the left of Figure 2. We found that the amplitude of the leading internal solitary wave is only 0.09358m. Then, we calculate the profile of internal solitary wave with $a = 0.09358m$ from KdV theory. we compared GN results with the KdV theory, which is shown on the right of Figure 2.

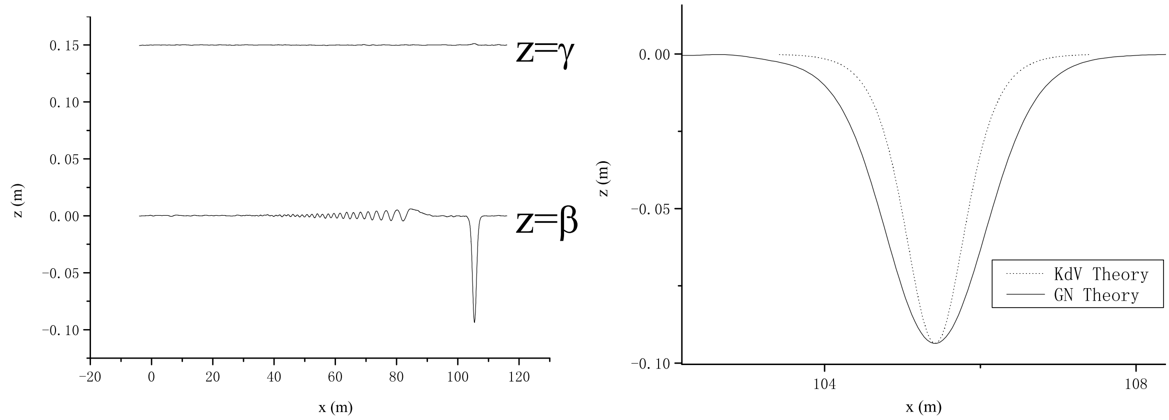

Figure 2: Snapshot at $t = 550$s with initial amplitude $a = 0.91h_2$ (left), comparison with KdV results (right)

We can see there are much difference between them. We tried to increase the amplitude of initial wave at $t = 0$, and we want to get a steady internal solitary wave with $a = 0.91h_2 = 0.1365$m. But, we failed. Because we use the KdV theory as the initial shape and velocity. As we know, the KdV theory can not predict large internal wave accurately.

We use another method to generate a solitary wave with amplitude $= 0.91h_2$. This method is similar to the physical experiment by Grue *et al.* (1999). The initial shape is shown on Figure 3. Here, the internal solitary wave is generated by gravity collapse. At $t = 0$s, the velocity is zero at everywhere. By careful changing the initial shape, finally we get a internal solitary wave with amplitude $a = 0.135$m.

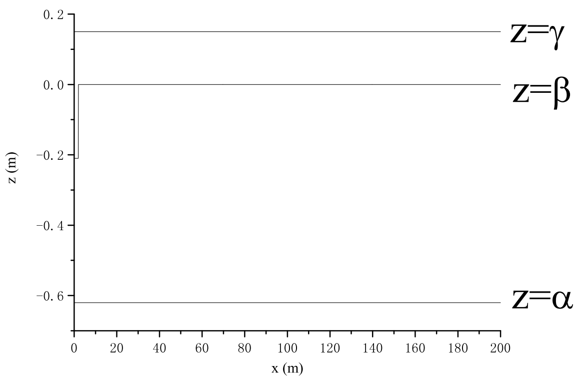

Figure 3: Initial wave profile similar to the physical experiment by Grue *et al.* (1999)

At $t = 355$s, the snapshot of the wave train is shown on the left of Figure 4. The amplitude of the leading solitary wave is 0.135m. Grue *et al.* (1999) gives the wave profile with amplitude $= 0.1365$m. The comparison between GN results and their fully nonlinear results is shown on the right of Figure 4.

We can see that the GN results are almost the same as the fully nonlinear model results by Grue *et al.* (1999). It should be noted that all cases here are simulated with $K^U = 1$ and $K^L = 1$. Higher level results will be presented at the workshop.

5 Conclusions

In this paper, the GN equations for two-layer fluid are given. The GN equations for two-layer flow has the similar form with one-layer case. We give some numerical results by using GN theory with $K^U = 1$ and

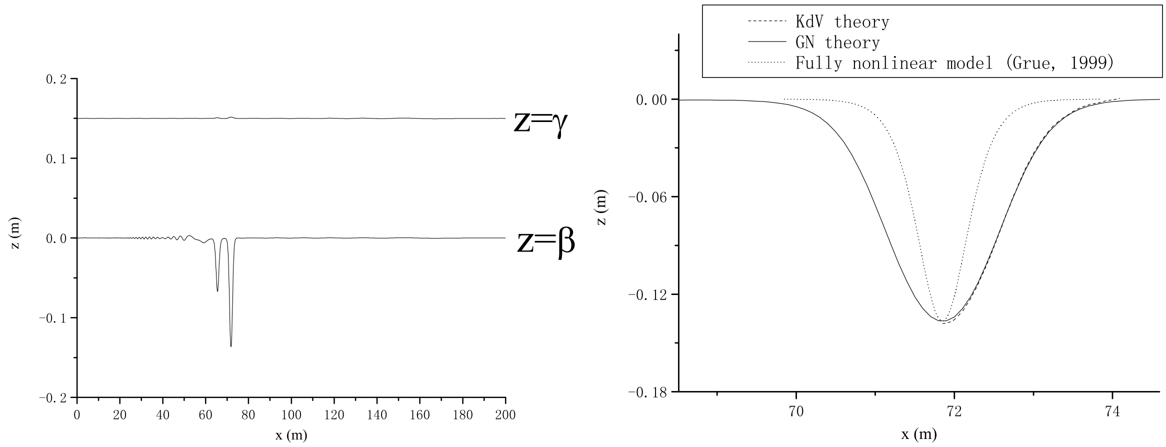

Figure 4: Snapshot at $t = 355s$ (left) and comparison with fully nonlinear model (right)

$K^L = 1$. For small amplitude internal solitary wave, the initial shape and velocity from KdV theory can be used as the incoming boundary condition in GN model. But, for large amplitude case, we need to find another method instead of the KdV theory. We use the similar method used by Grue *et al.* (1999) in their physical experiments. The internal solitary wave is generated by gravity collapse. For internal solitary wave amplitude $= 0.91h_2$, the GN theory gives the rather good results comparing with those by the fully nonlinear model in Grue *et al.* (1999).

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