WAVE FORCES ON OSCILLATING HORIZONTAL CYLINDER SUBMERGED UNDER NON-HOMOGENEOUS SURFACE

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1. Introduction

In a linear treatment, the problem of oscillations of a submerged body under a free surface and the resulting hydrodynamic loads have been thoroughly studied. A wide range of mathematical techniques was given by Linton & McIver (2001) for the solution of problems involving the interaction of waves with structure. Extensive bibliographical notes were made in this book. However, to the author's knowledge, there is no study on wave structure interaction problems in fluid having mixed boundary conditions on the upper surface.

In this paper, the linear 2-D water-wave problem describing small oscillations of a horizontal cylinder is considered. The surface of an ideal and incompressible fluid of finite depth is partly covered by a semi-infinite thin elastic plate with a free edge. The solution is written as a distribution of mass sources over the surface of the cylinder and an integral equation is applied for the unknown source strength. Appropriate Green's function is introduced using the method of matched eigenfunction expansions in much the same manner as in paper by Sahoo *et al.* (2001). Generation of flexural gravity waves by a submerged cylinder under infinite elastic plate was considered by Sturova (2011).

2. Statement of the problem

Let a Cartesian coordinate system be taken with the x-axis directed along the undisturbed upper boundary of the fluid perpendicular to the cylinder axis, and the y-axis pointing vertically upwards. The fluid is assumed to be inviscid and incompressible; its motion is irrotational. A semi-infinite elastic plate floats on the surface of the fluid (Figure 1). The surface of the fluid that is not covered with the plate is free. The plate draft is ignored. The fluid depth is equal to H. The wave motions are generated in the fluid by the small oscillations of submerged rigid body with wetted surface S at a frequency ω with amplitudes ζ_j $(j = 1, 2, 3)$ for the sway, heave and roll problems, respectively.

Under the usual assumptions of linear theory, the time-dependent velocity potential can be written as

$$\Phi(x, y, t) = \Re\left[i\omega \sum_{j=1}^{3} \zeta_j \varphi_j(x, y) \exp(i\omega t)\right],$$

where $\varphi_j(x, y)$ are complex valued functions and t is time. The radiation potentials $\varphi_j(x, y)$ satisfy the Laplace equation in the fluid domain

$$\nabla^2 \varphi_j = 0 \quad (-\infty < x < \infty,\ -H < y < 0) \tag{1}$$

except in the region occupied by the cylinder.

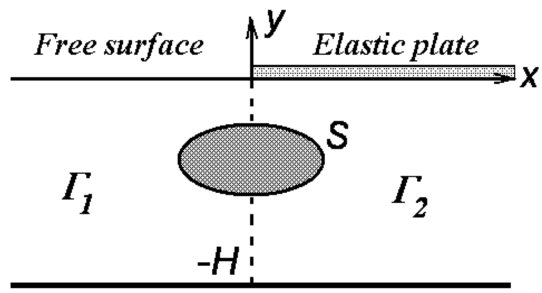

Figure 1: Schematic diagram.

The free surface condition in the open water region is given by

$$g\partial\varphi_j/\partial y - \omega^2 \varphi_j = 0 \quad (-\infty < x < 0,\ y = 0), \tag{2}$$

where g is the acceleration due to gravity. It is assumed that the plate is a thin homogeneous elastic material with uniform mass density ρ_s and thickness d. The plate is in contact with the water at all points for all time. On the elastic covered surface, the radiation potentials $\varphi_j(x, y)$ satisfy the boundary condition in the form

$$\left(D\frac{\partial^4}{\partial x^4} - \omega^2 M + g\rho\right)\frac{\partial \varphi_j}{\partial y} - \rho\omega^2 \varphi_j = 0 \tag{3}$$

$$(0 < x < \infty,\ y = 0),$$

where $D = Ed^3/[12(1 - \nu^2)]$, $M = \rho_s d$, E is the Young's modulus for the elastic plate, ν is its Poisson's ratio, ρ is the fluid density. At the plate edge, free edge boundary conditions require vanishing bending moment and shear force:

$$\frac{\partial^3\phi_j}{\partial x^2\partial y} = \frac{\partial^4\phi_j}{\partial x^3\partial y} = 0 \quad (x = 0+,\ y = 0). \qquad (4)$$

The boundary condition on the closed smooth contour of the submerged body S has the form:

$$\partial\varphi_j/\partial n = n_j \quad (x, y \in S). \qquad (5)$$

Here, $\mathbf{n} = (n_x, n_y)$ is the inward normal to the contour S. The notations

$$n_1 = n_x, \quad n_2 = n_y, \quad n_3 = (y-y_0)n_1-(x-x_0)n_2$$

are used where x_0, y_0 are the coordinates of the center of the roll oscillations.

The boundary condition at the bottom is

$$\partial\varphi_j/\partial y = 0 \quad (-\infty < x < \infty,\ y = -H). \qquad (6)$$

In the far field a radiation condition should be imposed that requires the radiated waves to be outgoing.

3. Method of solution

In order to solve the boundary-value problem (1)-(6), we introduce an unknown mass-source distribution $\sigma_j(x, y)$ over the contour S. We can now represent the radiation potentials at any point of the fluid in the form

$$\varphi_j(x, y) = \int_S \sigma_j(\xi, \eta)G(x, y; \xi, \eta)ds. \qquad (7)$$

Here, $G(x, y; \xi, \eta)$ is the Green function of the problem, which determines the velocity potential initiated by an oscillating mass source with unit strength, where (x, y) is the field point and (ξ, η) is the source point. The Green function must satisfy the following equation

$$\nabla^2 G = 2\pi\delta(x-\xi)\delta(y-\eta)$$

with the boundary conditions analogous to (2)-(4), (6) and the radiation condition in the far field, and δ is the Dirac delta-function.

In order to obtain the solution for the Green function, the eigenfunction expansion-matching method is employed. The fluid domain is divided into two regions: the downstream open region Γ_1 $(-\infty < x < 0,\ -H < y < 0)$ and the upstream plate-covered region Γ_2 $(0 < x < \infty,\ -H < y < 0)$. The solution for the Green function depends significantly on the position of the source point.

Case 1: source in Γ_1 $(\xi < 0)$.

In this case the value of $G(x, y; \xi, \eta)$ in Γ_i is denoted by $G_i^{(1)}(x, y; \xi, \eta)$ $(i = 1, 2)$. These functions will be sought as expansions in terms of eigenfunctions of corresponding boundary value problems:

$$G_1^{(1)} = G_0^{(1)} + R_0^{(1)}e^{ik_0x}\psi_0(y)+$$

$$\sum_{m=1}^{\infty} R_m^{(1)}e^{k_mx}\psi_m(y) \quad (x < 0), \qquad (8)$$

$$G_2^{(1)} = T_0^{(1)}e^{-ip_0x}f_0(y)+$$

$$\sum_{\substack{n=-2\\ n\neq 0}}^{\infty} T_n^{(1)}e^{-p_nx}f_n(y) \quad (x > 0), \qquad (9)$$

where

$$\psi_0 = \cosh k_0(y + H)/\cosh k_0H,$$

$$\psi_m = \cos k_m(y + H)/\cos k_mH \ (m = 1, 2, 3, ...),$$

$$f_0 = \cosh p_0(y + H)/\cosh p_0H,$$

$$f_n = \cos p_n(y+H)/\cos p_nH \ (n = -2, -1, 1, 2, 3, ...).$$

The constants k_m's satisfy the dispersion relations

$$\mathcal{K} = k_0 \tanh k_0H =$$

$$-k_m \tan k_mH \quad (m = 1, 2, 3, ...), \quad \mathcal{K} = \omega^2/g$$

with $(m-1)\pi/H < k_m < m\pi/H$ $(m = 1, 2, 3, ...)$. The constants p_n's satisfy the dispersion relations

$$K = p_0(1 + Lp_0^4)\tanh p_0H =$$

$$-p_n(1 + Lp_n^4)\tan p_nH \ (n = -2, -1, 1, 2, 3, ...)$$

with $L = D/(g\rho - \omega^2 M)$ and $K = \rho\omega^2/(g\rho - \omega^2 M)$. It should be noted that p_{-2} and p_{-1} are complex conjugates with positive real parts, p_n's are positive and real with $(n-1)\pi/H < p_n < n\pi/H$ $(n = 1, 2, 3, ...)$, and $R_m^{(1)}$, $T_n^{(1)}$ are unknown constants to be determined to obtain the Green function completely. The function $G_0^{(1)}(x, y; \xi, \eta)$ is a velocity potential due to a source submerged under infinite free surface

$$G_0^{(1)} = \ln\frac{r}{r_1} + pv\int_0^\infty F_1(y, \eta; k)\frac{\cos k(x-\xi)}{Z_1(k)}dk-$$

$$i\pi F_1(y, \eta; k_0)\frac{\cos k_0(x-\xi)}{Z_1'(k_0)}, \qquad (10)$$

where pv indicates the principal-value integration,

$$r^2 = (x-\xi)^2+(y-\eta)^2, \quad r_1^2 = (x-\xi)^2+(y+\eta)^2,$$

$$F_1 = \frac{2}{k(1+e^{-2kH})}\{[(k\cosh k\eta + \mathcal{K}\sinh k\eta)e^{-ky} -$$

$$(\mathcal{K}+k)e^{ky}\sinh k\eta]e^{-2kH} + ke^{k(y+\eta)}\},$$

$$Z_1(k) = \mathcal{K} - k\tanh kH, \quad Z_1'(k_0) \equiv dZ_1/dk|_{k=k_0}.$$

Because the velocity and pressure are continuous across the boundary between the regions Γ_1 and Γ_2, the full solution is obtained from matching conditions

$$\partial G_1^{(1)}/\partial x|_{x=0-} = \partial G_2^{(1)}/\partial x|_{x=0+},$$

$$G_1^{(1)}|_{x=0-} = G_2^{(1)}|_{x=0+} \quad (-H < y < 0).$$

Truncating the infinite series in (8), (9) and using the inner products as in the paper by Sahoo *et al.* (2001), unknown constants $R_m^{(1)}$ and $T_n^{(1)}$ can be determined.

Case 2: source in Γ_2 $(\xi > 0)$.

The Green function $G(x,y;\xi,\eta)$ in Γ_i is denoted by $G_i^{(2)}(x,y;\xi,\eta)$ and in the corresponding regions is expressed as

$$G_1^{(2)} = R_0^{(2)}e^{ik_0x}\psi_o(y) + \sum_{m=1}^{\infty} R_m^{(2)}e^{k_mx}\psi_m(y) \ (x<0),$$

$$G_2^{(2)} = G_0^{(2)} + T_0^{(2)}e^{-ip_0x}f_0(y) +$$

$$\sum_{\substack{n=-2\\ n\neq 0}}^{\infty} T_n^{(2)}e^{-p_nx}f_n(y) \quad (x>0), \qquad (11)$$

where the function $G_0^{(2)}(x,y;\xi,\eta)$ is a velocity potential due to a source submerged under infinite elastic plate

$$G_0^{(2)} = \ln\frac{r}{r_1} + pv\int_0^{\infty} F_2(y,\eta;p)\frac{\cos p(x-\xi)}{Z_2(p)}dp -$$

$$i\pi F_2(y,\eta;p_0)\frac{\cos p_0(x-\xi)}{Z_2'(p_0)}, \qquad (12)$$

where

$$F_2 = \frac{2}{p(1+e^{-2pH})}\{(p(Lp^4+1)\times$$

$$[(e^{-py}\cosh p\eta - e^{py}\sinh p\eta)e^{-2pH} + e^{p(y+\eta)}] -$$

$$2Ke^{-2pH}\sinh p\eta\sinh py\},$$

$$Z_2(p) = K - p(1+Lp^4)\tanh pH,$$

$$Z_2'(p_0) \equiv dZ_2/dp|_{p=p_0}.$$

Using boundary condition (5) on the body surface S, we obtain the integral equation for the functions $\sigma_j(x,y)$

$$\pi\sigma_j(x,y) - \int_S \sigma_j(\xi,\eta)\frac{\partial G}{\partial n}ds = n_j.$$

Once the distribution of the singularities $\sigma_j(x,y)$ has been calculated, we can determine the radiation potentials (7).

The radiation load acting on the oscillating body is determined by the force $\mathbf{F} = (F_1, F_2)$ and the moment F_3 which, without account for the hydrostatic term, have the form

$$F_k = \sum_{j=1}^{3}\zeta_j\tau_{kj} \quad (k=1,2,3),$$

$$\tau_{kj} = \rho\omega^2\int_S \varphi_j n_k ds = \omega^2\mu_{kj} - i\omega\lambda_{kj},$$

where μ_{kj} and λ_{kj} are the added mass and damping coefficients, respectively. There is the symmetry condition $\tau_{kj} = \tau_{jk}$.

4. Numerical results

Let us consider the most simple particular case of the problem when the elastic plate is substituted for the grid lid. In this case, boundary condition (4) is replaced by non-flow condition

$$\partial\varphi/\partial y = 0 \quad (0 < x < \infty,\ y = 0).$$

The solution for the Green's function $G_0^{(2)}$ can be obtained from (10) putting $\mathcal{K} = 0$. In addition the integral in (12) is taken in the ordinary sense and there is no term due to residue. The terms with p_{-2}, p_{-1}, p_0 are lacking in series (9), (11) and $p_n = n\pi/H$ $(n = 1,2,3,...)$. The calculations are performed for the circular contour $S : (x-c)^2 + (y+h)^2 = a^2$, where a is the radius of the circle and the coordinates of its center are equal to $x = c,\ y = -h$ $(h > 0)$.

It is well known, that non-zero values of the wave forces have only τ_{11} and τ_{22} for radiation by a submerged circular cylinder under infinite free surface. At the oscillations of the circular cylinder under infinite grid lid, non-zero values have only μ_{11} and μ_{22} and these values do not depend on frequency ω.

However, more complicated behavior of the wave forces takes place at mixed boundary conditions. Figures 2 and 3 give respectively dimensionless values of added mass and damping coefficients: $\bar{\mu}_{ij} = \mu_{ij}/(\pi\rho a^2)$, $\bar{\lambda}_{ij} = \lambda_{ij}/(\pi\rho a^2\omega)$ for the circular cylinder as functions of dimensionless frequency parameter $\bar{\omega}^2 = \omega^2 a/g$.

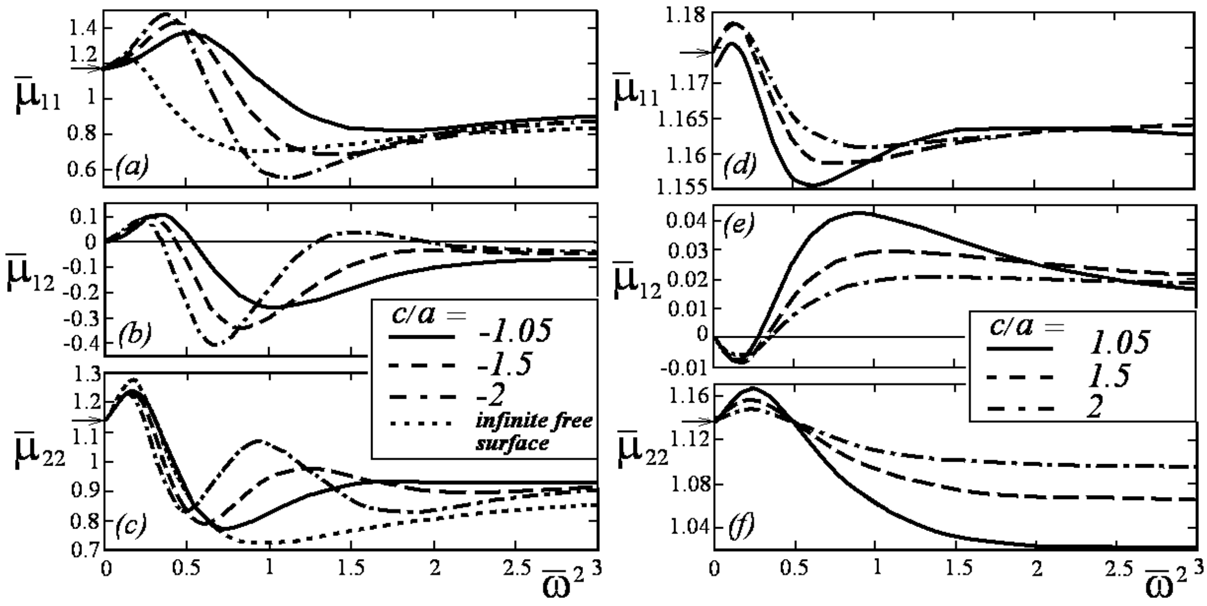

Figure 2: The added mass coefficients of a circular cylinder.

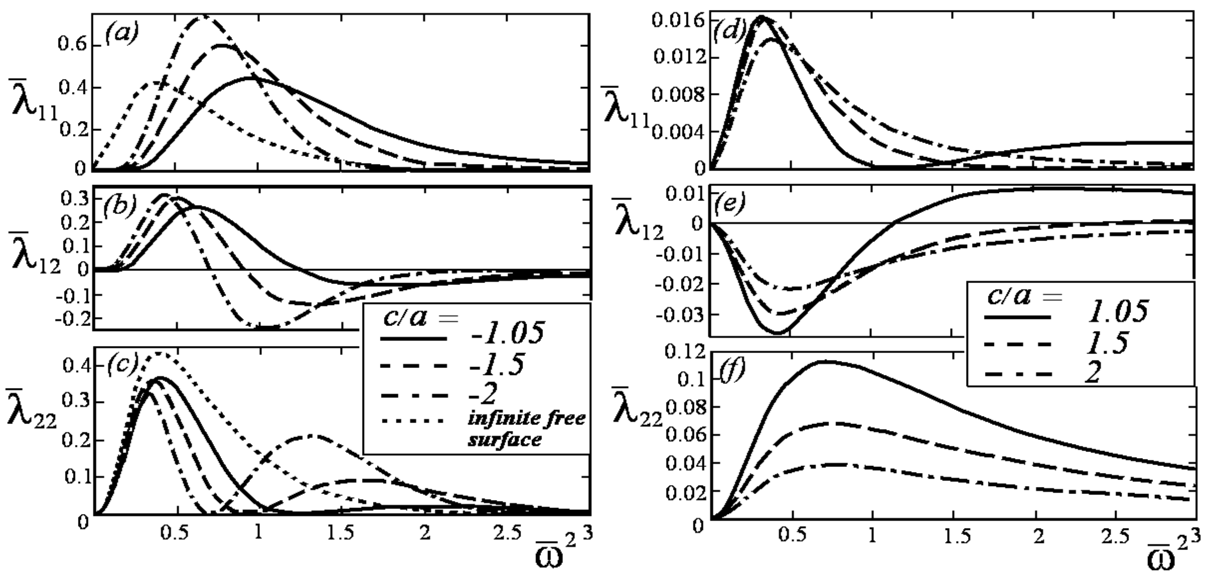

Figure 3: The damping coefficients of a circular cylinder.

The cylinder is submerged at $h = 2a$ and the depth of fluid is equal to $H = 10a$. Figures $2a, b, c$ and $3a, b, c$ give the values of wave forces for the cylinder submerged under the free surface, whereas figures $2d, e, f$ and $3d, e, f$ correspond to the position of cylinder under the grid lid. In figures $2a, c, d, f$, the horizontal arrows indicate the values of the added mass coefficients $\bar{\mu}_{11} \approx 1.1743$ and $\bar{\mu}_{22} \approx 1.1361$ for infinite grid lid. The dotted lines in figures $2a, c$ and $3a, c$ represent the values of added mass and damping coefficients for the circular cylinder submerged under infinite free surface, respectively.

More detailed results for the hydrodynamic load on the cylinder will be presented at the Workshop.

References

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