Hydrodynamic impact of three-dimensional bodies on waves.

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1) Introduction

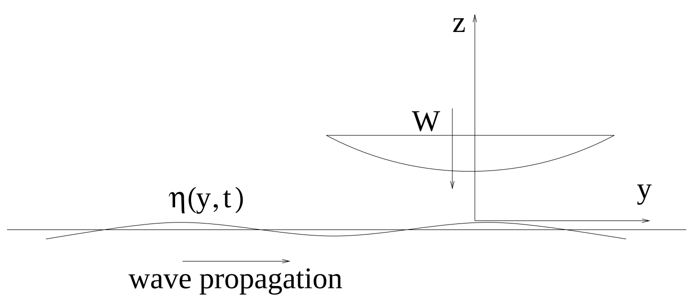
We consider a simple three-dimensional body entering vertically a liquid surface on which a regular wave propagates. As a simple body, an elliptic paraboloid is chosen. It is defined by two curvature radii at its initial contact point. The Wagner problem is posed in terms of the displacement potential and Galin's theorem is used to provide an analytical solution. Comparisons are done with experimental results.

2) Boundary value problem

The linearized boundary value problem (BVP) is formulated in terms of the displacement potential ϕ

$$\begin{cases} \Delta\phi = \phi_{,xx} + \phi_{,yy} + \phi_{,zz} = 0 & z < 0 \\ \phi = 0 & z = 0,\ (x,y) \in \mathrm{FS}(t) \\ \phi_{,z} = -h(t) + f(x,y) - \eta(y,t) & z = 0,\ (x,y) \in \mathrm{D}(t) \\ \phi \to 0 & (x^2+y^2+z^2) \to \infty, \end{cases} \tag{1}$$

where the regions FS(t) and D(t) are disconnected parts of the plane $z = 0$ and correspond to the free surface and the wetted area of the body, respectively. A closed curve, which separates the regions FS(t) and D(t), is denoted $\Gamma(t)$ and is referred to as the contact line. The body shape is represented by the equation $z = f(x,y)$, where $f(x,y)$ is a smooth positive shape function, $h(t)$ is the penetration depth of the body into the liquid and $\eta(y,t)$ represents the propagating wave along the y axis as illustrated below



The draft of the elliptic paraboloid is H. Its equation in a local coordinate system (X,Y,Z) attached to the body is

$$f(X,Y) = \frac{X^2}{2R_x} + \frac{Y^2}{2R_y} \tag{2}$$

where (R_x, R_y) are the curvature radii at the initial contact point. The wave is described in a fixed coordinate system attached to earth

$$\eta(y,t) = C\left(\cos(\kappa y - \omega t + \theta) - 1\right) \tag{3}$$

This is an Airy wave of amplitude C propagating along the y axis. By tuning the phase θ, we can manage to get a crest at time $t = 0$ and at the origin $y = 0$. We consider the first instant of penetration and duration of penetration (until separation of the flow at the top of the shape) is much smaller than the period of the wave $T = \frac{2\pi}{\omega}$. The circular frequency ω is related to the wave number κ by using the dispersion relation in infinite depth $\omega^2 = g\kappa$, with $g = 9.81 m/s^2$. We first choose the phase $\theta = 0$ and we expand the shape of the wave for small time and around the origin of the coordinate system, yielding

$$\eta(y,t) = \frac{1}{2R_v}y^2 - \frac{V_\varphi}{R_v}ty + \frac{V_\varphi^2}{2R_v}t^2 + \textit{higher order terms} \tag{4}$$

where $R_v = \frac{1}{C\kappa^2}$ is the curvature radius of the wave and $V_\varphi = \frac{\omega}{\kappa}$ is the phase velocity. The Neumann condition of BVP (1) reduces to

$$\phi_{,z} = -h(t) + \frac{x^2}{2R_x} + \frac{y^2}{2R_y} + \frac{y^2}{2R_v} - \frac{V_\varphi}{R_v} ty + \frac{V_\varphi^2}{2R_v} t^2 \tag{5}$$

The equivalent curvature radius $\tilde{R}_y$ is introduced

$$\tilde{R}_y = \frac{R_v R_y}{R_v + R_y} \tag{6}$$

It is worth noting that new curvature radius might be smaller than R_x, even though the radius R_y is originally greater than R_x. As an example if $R_y = 2m$ and $R_x = 0.75m$, the inequality $\tilde{R}_y < R_x$ holds as soon as $R_v < 1.2m$. We use the following change of variables

$$\tilde{x} = x, \quad \tilde{y} = y - \frac{\tilde{R}_y V_\varphi t}{R_v}, \quad \tilde{h} = h - \frac{1}{2} \frac{V_\varphi^2}{R_v + R_y} t^2 \tag{7}$$

with the help of which the Neumann condition can be arranged as follows

$$\phi_{,z} = -\tilde{h}(t) + \frac{\tilde{x}^2}{2R_x} + \frac{\tilde{y}^2}{2\tilde{R}_y} \tag{8}$$

which is a canonical form in order to apply Galin's theorem as described in Scolan and Korobkin (2012). The boundary problem with the condition (8) is equivalent to the problem of impact of a fictitious elliptic paraboloid onto initially flat free surface without waves. The fictitious paraboloid is different from the original one by the radius of curvature in y-direction and increased vertical displacement of the body. It should be noted that $\tilde{h}(t)$ contains an additional contribution $-\frac{1}{2}\frac{V_\varphi^2}{R_v+R_y}t^2 = -\frac{1}{2}Gt^2$ which decreases the vertical acceleration. The dissymmetry of the entry due to the propapating wave appears in the translational motion $\frac{\tilde{R}_y V_\varphi}{R_v} t = vt$.

3) Expansion of the wetted surface

The displacement potential reads

$$\phi(x, y, t) = -\frac{2\tilde{h}b}{3E(e)} \left(1 - \frac{\tilde{x}^2}{a^2} - \frac{\tilde{y}^2}{b^2}\right)^{3/2} \tag{9}$$

where implicitely the wetted surface is elliptic with aspect ratio k and (a, b) are the lengths of its major and minor semi axes respectively. The following identities are useful

$$2\tilde{h}\dot{b} = \dot{\tilde{h}}b, \qquad \frac{\dot{b}}{b} = \frac{\dot{a}}{a} \tag{10}$$

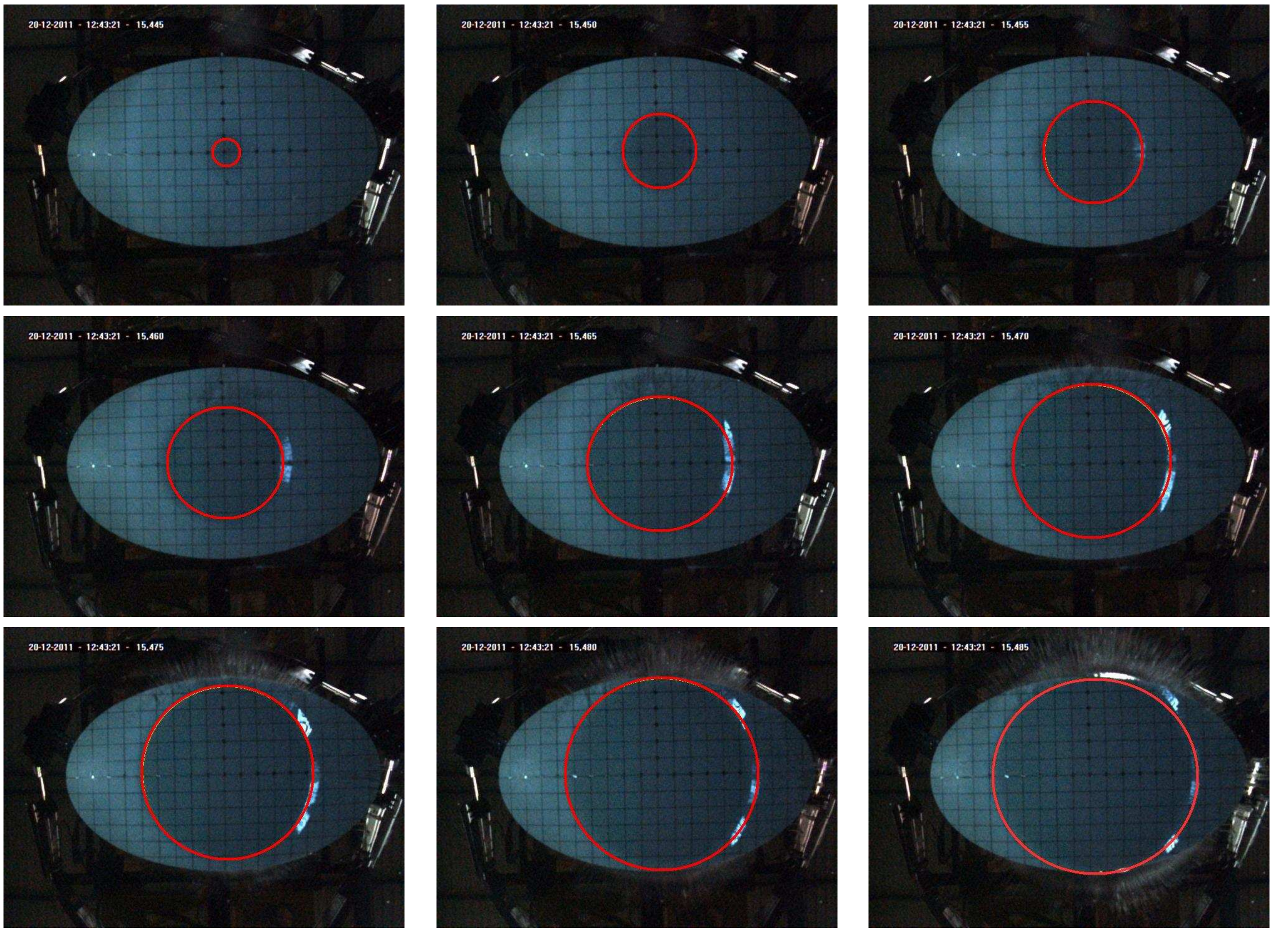
They follow from the application of Galin's theorem and the fact that $\frac{\tilde{R}_y}{R_x}$ is a constant in time. The relations between the sizes of the wetted surface and the data of the problem are the following

$$b = ka = \sqrt{2\tilde{R}_y\tilde{h}\left(2 - k^2\frac{D}{E}\right)}, \quad a = \sqrt{2R_x\tilde{h}\left(1 + k^2\frac{D}{E}\right)} = a_o\sqrt{\tilde{h}} \tag{11}$$

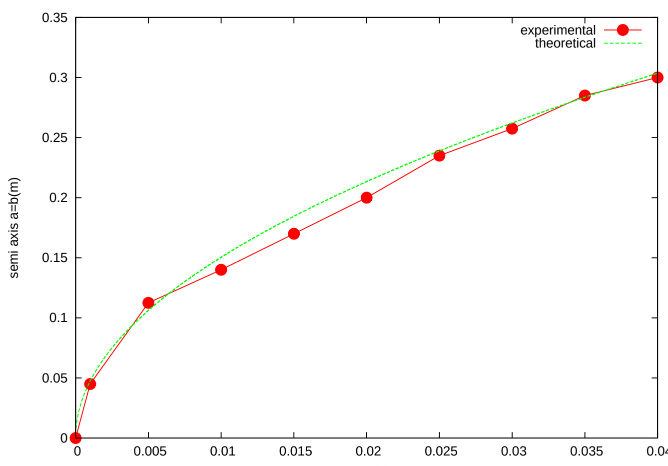
where $K(e)$, $E(e)$ and $D(e) = (K(e) - E(e))/e^2$ are the standard Elliptic Integrals. Those are functions of the eccentricity $e = \sqrt{1 - k^2}$ only. The aspect ratio k is given by

$$k_\gamma^2 = \sqrt{\frac{\tilde{R}_y}{R_x}} = k^2 \frac{1 + k^2 D/E}{2 - k^2 D/E} \tag{12}$$

In order to validate these results, we use the experimental data base obtained during the campaign described in Scolan (2012). A regular wave is generated with amplitude $C = 0.048m$ and period $T = 0.98s$. The curvature radius is $R_v = 1.186m$. The corrected curvature radius appearing after asymptotic expansion is $\tilde{R}_y \approx 0.745m$ quite similar to the curvature radius along the x direction which is $R_x \approx 0.75m$. We expect that the wetted surface in expansion is circular. Among the instrumentation used in the set-up, an immersed camera records the expansion of the wetted surface at the sampling frequency $200Hz$. The obtained pictures are shown below for the initial stage of penetration.



To sum up the results, we compare in the figure below the time variations of the length of the semi axes a and b given by equations (11).



4) Local and global loads

The pressure is calculated from the second derivative in time of the displacement potential.

$$p = -\rho\phi_{,t^2} \tag{13}$$

After some algebra and by using the idendities (10), we arrive at

$$p = \frac{\rho}{E\sqrt{F}}\left(\dot{M}F + \frac{1}{2}M\dot{F}\right) \tag{14}$$

with

$$F = 1 - \frac{x^2}{a^2} - \frac{(y-vt)^2}{b^2}, \qquad M = 2\tilde{h}\left(\dot{b} + \frac{v(y-vt)}{b}\right) \tag{15}$$

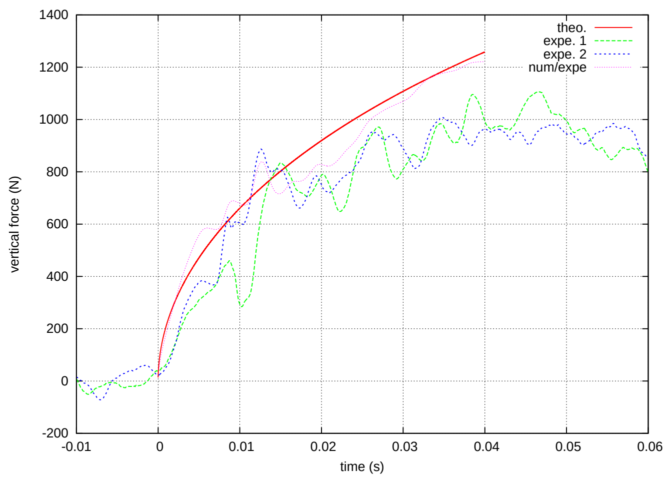
The first order force follows from the integration of the pressure on the wetted surface. However the boundary condition on the free surface and the continuity of the displacement potential (and its derivatives) through the contact line, lead to the following expression of the vertical force

$$F_z = -\rho\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\int_{D(t)}\phi\,\mathrm{d}x\mathrm{d}y \tag{16}$$

By using (9), it is finally obtained

$$F_z = \frac{4\pi\rho}{15E}k^2a_o^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\tilde{h}^{5/2}\right] \tag{17}$$

The time varitaion of F_z is plotted below.



As usual the first order approximation overpredicts the force. If the experimental velocity and acceleration are introduced in the theoretical force formulation, it is observed the influence of the actual kinematics. Before separation occuring at $t \approx 0.04s$, the error is about 20 %. The Modified Logvinovich Model of the pressure must hence be used. Then the numerical integration of MLM pressure reduces the discrepancy.

5) References

1. Wagner, H. 1932 Über Stoss- und Gleitvorgänge an der Oberfläche von Flüssigkeiten. *ZAMM* **12**, pp. 193-215.

2. Scolan Y.-M. & Korobkin A. A., 2012, Hydrodynamic impact (Wagner) problem and Galin's theorem. 27^{th} International Workshop on Water Waves and Floating Bodies, Copenhagen, Denmark.

3. Scolan Y.-M., 2012, Hydrodynamic loads during impact of a three dimensional body with an arbitrary kinematics (in french). Proc. 13^{th} Journées de l'Hydrodynamique, Chatou, France.