Trapped modes in an ice-covered two-layer fluid of finite depth

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1 Introduction

Recently there has been considerable interest in the investigation of ice-wave interaction problems due to an increase in scientific activities in the polar regions. In order to understand the mechanism and effects of wave propagation through the marginal ice zone in the polar regions, the ice-cover is modeled as a thin ice-sheet of which a very small part is immersed in water and is composed of materials having elastic properties. The flexural gravity wave propagation in a two-layer fluid has been investigated extensively which are substantiated by some notable works ([1], [2], [3]). To the best of the authors' knowledge, no investigation of flexural trapped waves in a two-layer fluid has taken place till date. In fact there has been relatively very few investigations on trapped modes for a two-layer fluid with free surface ([4], [5]). A trapped mode has finite energy, does not radiate waves to infinity, and will persist for all time. The significance of such a mode is that if, for a specified frequency of oscillation the structure does not support a trapped mode, then the solutions to the radiation and scattering problems at that frequency are unique.

In the present work, we consider a two-layer fluid of finite depth bounded above by a thin ice-cover and below by an impermeable horizontal bottom surface. The effect due to surface tension at the interface between the two fluids is neglected. Under the usual assumptions of linear water wave theory and by using Multipole expansion method we examine the existence of trapped mode when a horizontal, circular cylinder is placed in either of the layers.

2 Mathematical formulation

Cartesian coordinates are chosen with (x, y)- plane in the horizontal direction, z- axis in the vertically upward direction and $z = 0$ as the mean position of the interface. Each fluid is assumed to be of infinite horizontal extent in x- and y- directions. The potential flow theory ensures the existence of the velocity potential $\Phi(x, y, z, t)$ which satisfies Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad \text{in the fluid region.}$$

Let Φ^I be the potential in the upper fluid (of density ρ^I), $0 < z < d$ and Φ^{II} be the potential in the lower fluid (of density ρ^{II}), $-h < z < 0$. Denoting the ratio of the densities of the two fluids $\rho^I/\rho^{II}(< 1)$ by ρ, the linearized boundary conditions at the ice-cover, on the interface and at the bottom surface are:

$$\Big(D\nabla^4_{x,y} + 1 + \frac{\varepsilon}{g}\Big)\frac{\partial\Phi^I}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi^I}{\partial t^2} = 0 \qquad \text{on} \qquad z = d,$$

$$\frac{\partial\Phi^I}{\partial z} = \frac{\partial\Phi^{II}}{\partial z} \qquad \text{on} \qquad z = 0,$$

$$\rho\Big(\frac{\partial\Phi^I}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi^I}{\partial t^2}\Big) = \frac{\partial\Phi^{II}}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi^{II}}{\partial t^2} \qquad \text{on} \qquad z = 0,$$

$$\frac{\partial\Phi^{II}}{\partial z} = 0 \qquad \text{on} \qquad z = -h,$$

where $D = L/(\rho^I g)$; L is the flexural rigidity of the elastic ice-cover, $\varepsilon = (\rho_0/\rho^I)h_0$; ρ_0 is the density of the ice, h_0 is the very small thickness of the ice-cover.

It is assumed that the fluid motion is simple harmonic in time with angular frequency ω, so the velocity potential can be expressed as $\Phi^m(x, y, z, t) = Re[\phi^m(x, z)e^{ily}e^{-i\omega t}]$, $m = I, II$. In this case each $\phi^m(x, z)$ satisfies Helmholtz equation $(\nabla^2_{x,z} - l^2)\phi^m = 0$.

Within this framework progressive waves take the form (up to an arbitrary multiplicative constant)

$$\phi^I = exp(\pm ix\sqrt{u^2 - l^2})\Big(F_+(u)e^{u(z-d)} + F_-(u)e^{-u(z-d)}\Big) \quad \text{and} \quad \phi^{II} = exp(\pm ix\sqrt{u^2 - l^2})\cosh u(z + h)F(u),$$

where

$$F_\pm(u) = \Big(Du^4 + 1 - \varepsilon K\Big)u \pm K \quad \text{and} \quad F(u) = \frac{F_+(u)e^{-ud} - F_-(u)e^{ud}}{\sinh uh},$$

with $K = \frac{\omega^2}{g}$, g being the acceleration due to gravity and u satisfying the dispersion relation

$$(u + K\sigma)F_+(u)e^{-2u(d+h)} + (u - K\sigma)F_-(u) - (u + K)F_-(u)e^{-2uh} - (u - K)F_+(u)e^{-2ud} = 0.$$

This equation has exactly two positive real roots λ_1 and λ_2. The detail analysis of the roots for dispersion relation is given in Bhattacharjee and Sahoo [1]. For the existence of trapped modes we require $\phi^I, \phi^{II}, |\nabla\phi^I|, |\nabla\phi^{II}| \to 0 \quad \text{as} \quad |x| \to \infty$ and hence we restrict l to be in the range $l > \lambda_2 > \lambda_1$, which ensures that no wave propagation to infinity takes place on the interface or near the ice-cover.

3 Solutions by multipoles

A horizontal circular cylinder of radius a having its axis along $z = f$ and its generator running parallel to the y-axis is placed in a two-layer fluid. If $f > 0$, the cylinder is in the upper fluid, whereas for $f < 0$ the cylinder is in the lower fluid. Polar coordinates (r, θ) are defined in the (x, z)- plane centered on $(0, f)$ as $x = r\sin\theta$ and $z = f - r\cos\theta$.

3.1 Cylinder submerged in the upper layer

Symmetric multipoles in the upper layer are defined by

$$\phi_n^I = K_n(lr)\cos n\theta + \int_0^\infty \cosh nu \cos(lx\sinh u)\Big[A_I(v)e^{vz} + B_I(v)e^{-vz}\Big]du,$$

with

$$\begin{aligned} A_I(v) &= \frac{F_+(v)e^{-2vd}}{F_-(v)}\Big(B_I(v) + (-1)^n e^{vf}\Big), \\ B_I(v) &= \frac{\Big((-1)^{n+1}F_+(v)e^{v(f-2d)} - F_-(v)e^{-vf}\Big)}{G(v)}\Big((v+K\sigma)e^{-2vh} - v + K\Big), \end{aligned}$$

where

$$G(v) = (v+K\sigma)F_+(v)e^{-2v(d+h)} + (v-K\sigma)F_-(v) - (v+K)F_-(v)e^{-2vh} - (v-K)F_+(v)e^{-2vd}; \qquad v = l\cosh u. \quad (3.1)$$

Here $K_n(.)$ are modified Bessel functions of the second kind of order n and the integrals are Cauchy Principal Value integrals. $K_n(lr)\cos n\theta$ has the integral representation as given in [6]. The total velocity potential can now be written as [5]

$$\phi = \sum_{m=0}^{\infty} \alpha_m \phi_m^I, \qquad \text{with} \qquad \phi_m^I = K_m(lr)\cos m\theta + \sum_{n=0}^{\infty} A_{mn} I_n(lr)\cos n\theta,$$

where $I_n(.)$ are modified Bessel functions of the first kind of order n and

$$A_{mn} = \epsilon_n \int_0^\infty \cosh mu \cosh nu \Big[(-1)^n A_I(v)e^{vf} + B_I(v)e^{-vf}\Big]du, \qquad \text{with} \qquad \epsilon_0 = 1, \epsilon_n = 2, n \geq 1.$$

We note that since $v > l$, there will be no singularities of the integrand on the real axis. Applying the body boundary condition, $\frac{\partial\phi}{\partial r} = 0$ on $r = a$, we obtain an infinite system of linear equations in the unknowns α_m, which is

$$\alpha_n + \frac{I_n'(la)}{K_n'(la)} \sum_{m=0}^{\infty} \alpha_m A_{mn} = 0. \quad (3.2)$$

For a fixed geometrical configuration, the problem of finding the trapped mode frequencies is completely specified by the two non-dimensional parameters Ka and la: one of the parameters may be fixed and the other varied until the value of the determinant becomes approximately zero. Then using that specific value of Ka in the dispersion relation we get the values of two wavenumbers $\lambda_1 a$ and $\lambda_2 a$.

3.2 Cylinder submerged in the lower layer

We now consider the problem with the cylinder positioned in the lower fluid layer. The multipoles singular at $z = f(< 0)$ are required to be modified. This can be done in the same way as was done previously for the case of cylinder in the upper layer $f > 0$. The symmetric multipoles in the lower layer are

$$\phi_m^{II} = K_m(lr)\cos m\theta + \sum_{n=0}^{\infty} B_{mn} I_n(lr)\cos n\theta,$$

where

$$B_{mn} = \epsilon_n \int_0^\infty \cosh mu \cosh nu \Big[(-1)^n C_{II}(v)e^{vf} + D_{II}(v)e^{-vf}\Big]du,$$

$$C_{II}(v) = \frac{\Big((-1)^{n+1}e^{vf} - e^{-v(f+2h)}\Big)}{G(v)}\Big((v+K\sigma)F_+(v)e^{-2vd} - (v+K)F_-(v)\Big), \qquad D_{II}(v) = \Big(C_{II}(v) + e^{-vf}\Big)e^{-2vh},$$

where $G(v)$ is same as in Eq. (3.1).

Once again, the integrand has no singularities on the real axis. By applying the body boundary condition, $\frac{\partial \phi}{\partial r} = 0$ on $r = a$, we obtain the same type of system of equations like (3.2). Here also, as in the previous case, by fixing the other parameters and varying the frequencies Ka, we conveniently locate the zeros of the truncated determinant. It is already known that the values of those frequencies correspond to the trapped modes.

4 Numerical results

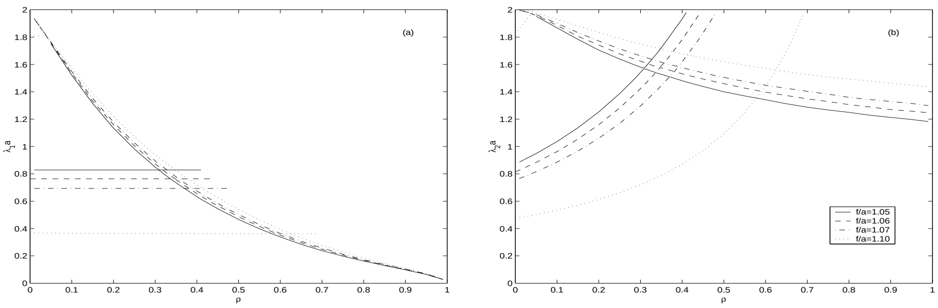

Figure 1: Trapped mode wavenumbers against ρ for a cylinder of radius a in the upper fluid layer for different submergence depths f/a; $la = 2$, $d/a = 2.1$, $h/a = 6$, $D/a^4 = 0.001$ and $\epsilon/a = 0.001$.

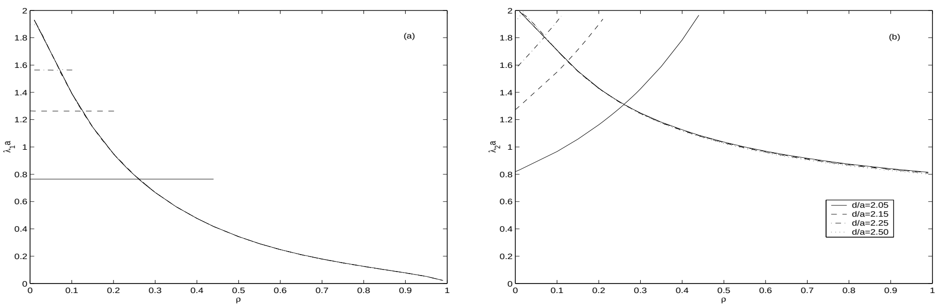

Figure 2: Trapped mode wavenumbers against ρ for a cylinder of radius a in the upper fluid layer for different depths d/a of the upper layer; $la = 2$, $f/a = 1.01$, $h/a = 6$, $D/a^4 = 0.001$ and $\epsilon/a = 0.001$.

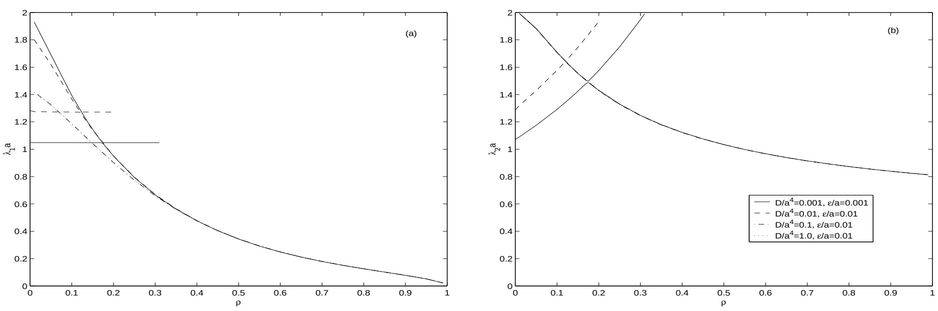

Figure 3: Trapped mode wavenumbers against ρ for a cylinder of radius a in the upper fluid layer for different ice-parameters; $la = 2$, $f/a = 1.01$, $d/a = 2.1$ and $h/a = 6$.

For numerical evaluation of the zeros of the determinant, we truncate the systems to a 32×32 system and the result presented here are obtained correct up to three decimal places. When the cylinder is placed in the upper layer, for each value of submergence depth there are at least two curves for each of the wavenumbers $\lambda_1 a$ and $\lambda_2 a$ (figure 1). For both the wavenumbers, with increase in density ratio, the first and second modes come very close to each other at near crossing points, after which the second modes terminate and each of the first modes decreases to some fixed value for corresponding submergence depth. For each submergence depth, these near crossing points have also been observed in [5] but the positions of these points get shifted downwards due to the presence of ice-cover. We have seen in [5] that third modes exist when the cylinder is nearer to the free surface but when the free surface gets replaced by a thin ice-cover then a third mode exists when the surface of the cylinder just touches the ice-cover. When an ice-cover replaces the free surface, the wavenumbers $\lambda_1 a$ and $\lambda_2 a$ of the trapped modes increase. With increase in depth of the upper layer, these near crossing points increase to $la = 2$ and when depth increases further, only first mode always exists and it will become independent of depth of the upper layer (figure 2). We have also plotted the trapped mode wavenumbers $\lambda_1 a$ and $\lambda_2 a$ against density ratio for four different sets of non-dimensionalised ice-parameters (figure 3). It is observed that with increase in flexural rigidity, the ice-cover behaves more like a rigid lid and for that there exists only one mode for both the wavenumbers which is very similar to the modes found in the rigid lid problem already taken up by the authors.

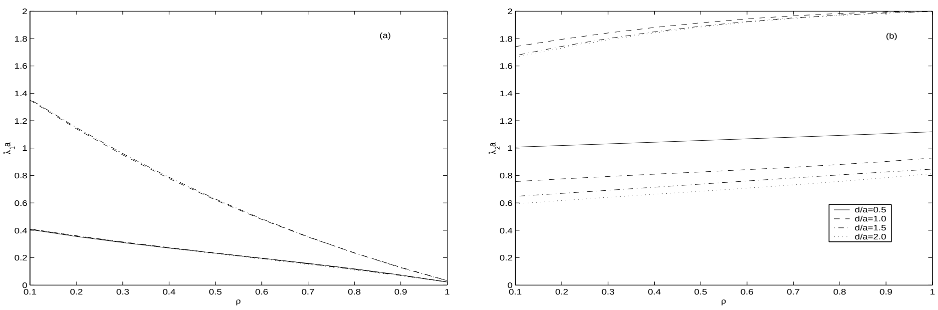

Figure 4: Trapped mode wavenumbers against ρ for a cylinder of radius a in the lower fluid layer for different depths d/a of the upper layer; $la = 2$, $h/a = 6$, $f/a = -1.01$, $D/a^4 = 0.001$ and $\epsilon/a = 0.001$.

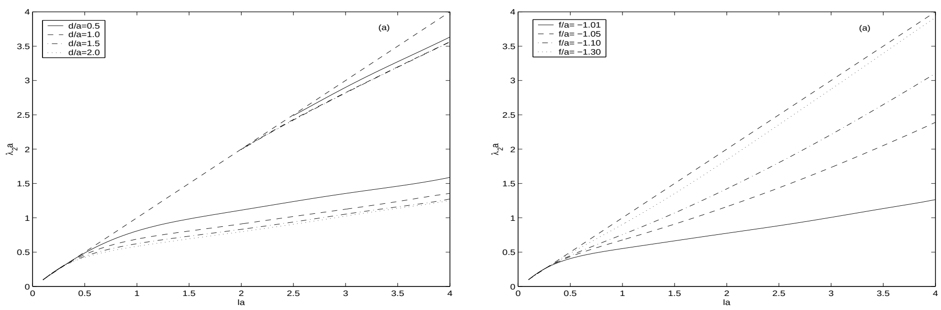

Figure 5: Dispersion curves for a cylinder of radius a in the lower layer; $\rho = 0.95$, $h/a = 6$, $D/a^4 = 0.001$ and $\epsilon/a = 0.001$ (a) for different depths d/a of the upper layer; $f/a = -1.01$ (b) for different submergence depths f/a; $d/a = 3$.

When the cylinder is submerged in the lower layer near the interface, $la = 2$, we observe that for depth d/a of the upper layer ≥ 1, there are two curves which correspond to the two modes for each of the wavenumbers $\lambda_1 a$ and $\lambda_2 a$. Otherwise, there exists only one mode (figure 4). For the wavenumber $\lambda_1 a$, these two modes tend to zero as $\rho \to 1$ for any depth of the upper layer. For wavenumber $\lambda_2 a$ as $\rho \to 1$, the first mode increases marginally to a fixed value corresponding to each depth of the upper layer and the second modes tend to $la = 2$. When we plot the trapped mode wavenumber $\lambda_2 a$ against la, we observe that with increase in depth of the upper layer, $\lambda_2 a$ decrease, the first mode being affected more than the second mode (figure 5(a)). From figure 5(b) we observe that as the cylinder approaches the interface, the curves fold out from the upper bound. Now if the cylinder is positioned in the upper layer, then also dispersion curves are found to have the same trend.

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