A novel connection between the Ursell and Dean vertical barrier potentials

R. Porter & D. V. Evans,
School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK

Abstract

This Abstract is dedicated to the memory of Professor Fritz Ursell whose influence continues to be reflected in contributions to the Workshop.

Fritz Ursell's first published paper in 1947 was on the scattering of an incident wave by a thin vertical barrier, aspects of which coincidentally also formed the basis of the first paper published by each of us, one or two generations later. This 1947 paper was remarkable in providing one of the few explicit solutions in linear water-wave theory, namely the potential everywhere in the fluid when an incident two-dimensional wave-train in deep water is scattered by a thin vertical rigid barrier immersed to a depth a. He also presented the solution to the complementary problem solved earlier by Dean (1945) who used complex function theory, in which the vertical barrier extends from infinity in the depths of the fluid to a point a distance a beneath the free surface.

In this paper we show that these distinct solutions and corresponding reflection coefficients can be related through two entirely new potentials which, given one of the above solutions enables the other to be determined. The results are a special case of a more general theory which can include finite depth, oblique incidence and different geometric configurations (Porter & Evans (2013, in preparation)).

1 The role of the new connection potentials

Cartesian coordinates are defined with the origin in the mean free surface and y pointing vertically downwards into the infinite depth fluid. A thin barrier occupies the interval $y \in B$ of the plane $x = 0$ and the gap in the barrier occupies the interval $y \in G$. In the Ursell and Dean problems B is the interval $(0, a)$ and (a, ∞) and G the interval (a, ∞) and $(0, a)$ respectively.

A plane two-dimensional monochromatic wave of radian frequency ω is incident from $x > 0$ on the barriers. Under the usual assumptions of linearised wave theory there exists a velocity potential $\Re\{\Phi(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ This complex-valued potential $\Phi(x, y)$ satisfies the equations

$$\Phi_{xx} + \Phi_{yy} = 0, \qquad y > 0, \tag{1}$$

with

$$K\Phi + \Phi_y = 0, \qquad \text{on } y = 0 \tag{2}$$

and $K = \omega^2/g$ where g is gravitational acceleration and

$$\nabla\Phi \to 0 \qquad y \to \infty. \tag{3}$$

We must also impose no-flow conditions on the barrier

$$\Phi_x(0^{\pm}, y) = 0, \qquad y \in B \tag{4}$$

and specify a radiation condition. We assume a wave incident from $x = \infty$ so that

$$\Phi(x, y) \sim \begin{cases} \mathrm{e}^{-\mathrm{i}Kx-Ky} + R\mathrm{e}^{\mathrm{i}Kx-Ky}, & x \to \infty \\ T\mathrm{e}^{-\mathrm{i}Kx-Ky}, & x \to -\infty \end{cases} \tag{5}$$

where R and T are the reflection and transmission coefficients. It can easily be shown that

$$\Phi(x, y) = \begin{cases} \mathrm{e}^{-\mathrm{i}Kx-Ky} + \mathrm{e}^{\mathrm{i}Kx-Ky} + \phi(x, y), & x > 0 \\ -\phi(-x, y), & x < 0 \end{cases} \tag{6}$$

where $\phi(x,y)$ is defined in $x>0$ and satisfies (1), (2) and (3) in addition to the boundary conditions

$$\phi_x(0,y)=0, \quad y\in B, \qquad \phi(0,y)+\mathrm{e}^{-Ky}=0, \quad y\in G \tag{7}$$

which results from imposing (4) on Φ and continuity of $\Phi(x,y)$ across $x=0$ when $y\in G$, based on the decomposition (6). Under this definition $\phi(x,y)\sim(R-1)\mathrm{e}^{\mathrm{i}Kx-Ky}$ as $x\to\infty$ with $T=R-1$. With r a local measure of the distance from any barrier edge immersed in the fluid, we also require $|\nabla\phi|\sim r^{-1/2}$ as $r\to 0$.

We label the Ursell and Dean potentials in (6) by $\phi^u(x,y)$ and $\phi^d(x,y)$ respectively, satisfying

$$\phi_x^{u/d}(0,y)=0, \quad y\in(0,a)/(a,\infty), \qquad \phi^{u/d}(0,y)+\mathrm{e}^{-Ky}=0, \quad y\in(a,\infty)/(0,a)$$
$$\phi^{u/d}(x,y)\sim(R^{u/d}-1)\mathrm{e}^{\mathrm{i}Kx-Ky} \quad \text{as } x\to\infty \tag{8}$$

Thus, from Ursell (1947) we have

$$\phi^u(x,y)+\mathrm{e}^{\mathrm{i}Kx-Ky}=C\left(\pi I_1(Ka)\mathrm{e}^{\mathrm{i}Kx-Ky}+\int_0^\infty\frac{L(k,y)J_1(ka)\mathrm{e}^{-kx}}{(k^2+K^2)}\mathrm{d}k\right) \tag{9}$$

where $C=(\pi I_1(Ka)+\mathrm{i}K_1(Ka))^{-1}$ and $R^u=\pi I_1(Ka)C$, and

$$\phi^d(x,y)+\mathrm{e}^{\mathrm{i}Kx-Ky}=B\left(K_0(Ka)\mathrm{e}^{\mathrm{i}Kx-Ky}-\int_0^\infty\frac{L(k,y)J_0(ka)\mathrm{e}^{-kx}}{(k^2+K^2)}dk\right) \tag{10}$$

where $B=(K_0(Ka)+\mathrm{i}\pi I_0(Ka))^{-1}$ and $R^d=K_0(Ka)B$. The above expressions involve Bessel functions whilst $L(k,y)=k\cos ky-K\sin ky$.

We now connect these potentials through the introduction of new potentials $\psi^{u/d}(x,y)$. Thus let

$$\phi^{u/d}(x,y)+\mathrm{e}^{\mathrm{i}Kx-Ky}=\mathrm{i}K^{-1}\left(\phi_x^{d/u}(x,y)+A^{d/u}\psi_x^{d/u}(x,y)\right). \tag{11}$$

Then from (8) we require

$$\psi_x^{u/d}(0,y)=0, \quad y\in(0,a)/(a,\infty), \qquad \psi_{yy}^{u/d}(0,y)=0, \quad y\in(a,\infty)/(0,a) \tag{12}$$

where the second condition arises from differentiating (11) with respect to x, using the Laplacian, and finally noting the relation $\phi_{yy}^{u/d}(0,y)=-K^2\mathrm{e}^{-Ky}$ for $y\in(a,\infty)/(0,a)$ from (8). We also assume

$$\psi^{u/d}(x,y)\sim\widetilde{R}^{u/d}\mathrm{e}^{\mathrm{i}Kx-Ky}, \qquad x\to\infty \tag{13}$$

where $\widetilde{R}^{u/d}\in\mathbb{C}$ is to be determined.

The constant $A^{d/u}$ is to be determined from the condition

$$\lim_{r\to0}r^{1/2}\left(\phi_x^{d/u}(x,y)+A^{d/u}\psi_x^{d/u}(x,y)\right)=0, \qquad \text{where } r=(x^2+(y-a)^2)^{1/2} \tag{14}$$

since $\phi^{u/d}$ is bounded near $r=0$.

The second condition in (12) may be integrated to give

$$\psi^d(0,y)=1-Ky, \quad y\in(0,a), \qquad \text{and} \qquad \psi^u(0,y)=1, \quad y\in(a,\infty) \tag{15}$$

to ensure that (2) is satisfied by $\psi^d(x,y)$ and (3) by $\psi^u(x,y)$.

Thus, we have shown in (11) that $\phi^{u/d}$ can be expressed in terms of the sum of the x-derivative of $\phi^{d/u}$ and a 'connection' potential $\psi^{d/u}(x,y)$ satisfying the same Neumann condition as $\phi^{d/u}(x,y)$ on the barrier but with different Dirichlet conditions described by (15). Using the far-field asymptotic form designated to each term in (11) and letting $x\to\infty$ gives

$$R^{u/d}=1-R^{d/u}-A^{d/u}\widetilde{R}^{d/u}. \tag{16}$$

2 Derivation of the connection potential $\psi^d(x,y)$.

As an illustration of the theory we shall derive the connection potential $\psi^d(x,y)$ from first principles and confirm that together with knowledge of the Dean potential and corresponding reflection coefficient it can be used through (11) and (16) to derive the Ursell potential and its reflection coefficient.

The most general potential satisfying (1), (2), (3) and (13) is written

$$\psi^d(x,y) = \widetilde{R}^d \mathrm{e}^{\mathrm{i}Kx-Ky} + \frac{2}{\pi}\int_0^\infty \frac{A(k)L(k,y)\mathrm{e}^{-kx}}{k(k^2+K^2)}\mathrm{d}k, \tag{17}$$

where $\widetilde{R}^d$ and $A(k)$ are unknowns. We define

$$U^d(y) \equiv \psi^d_x(0,y) = \mathrm{i}K\widetilde{R}^d\mathrm{e}^{-Ky} - \frac{2}{\pi}\int_0^\infty \frac{A(k)L(k,y)}{(k^2+K^2)}\mathrm{d}k \tag{18}$$

which is zero when $y > a$ on account of (12). Using Havelock's (1929) inversion theorem

$$\widetilde{R}^d = -2\mathrm{i}\int_0^a U^d(y)\mathrm{e}^{-Ky}\mathrm{d}y, \qquad \text{and} \qquad A(k) = -\int_0^a U^d(y)L(k,y)\mathrm{d}y \tag{19}$$

where use has been made of $U^d(y) = 0$ for $y > a$ to restrict the integration interval to $(0,a)$. It follows from substition of $A(k)$ from (19) into (17) and the imposition of (15) that

$$\int_0^a U^d(t)K(y,t)\mathrm{d}t = f(y), \quad y \in (0,a) \qquad \text{where } K(y,t) = \int_0^\infty \frac{L(k,t)L(k,y)}{k(k^2+K^2)}dk. \tag{20}$$

and with $f(y) = \frac{1}{2}\pi(\widetilde{R}^d\mathrm{e}^{-Ky} + Ky - 1)$. Ursell (1947) shows how this integral equation may be transformed after application of the differential operator $K + \partial/\partial y$ to become

$$\int_0^a \frac{V^d(t)}{y^2-t^2}\mathrm{d}t = -\left(f'(y) + Kf(y)\right)/y, \quad y \in (0,a) \qquad \text{where } V^d(y) = U^d(y) + K\int_a^y U^d(t)\mathrm{d}t. \tag{21}$$

Notice that $V^d(y)$ has the same singular behaviour as $U^d(y)$ near $y = a$ such that $\lim_{y\to a}(U^d(y) - V^d(y)) = 0$ and is bounded near $y = 0$.

For the particular $f(y)$ in this case $-(f'(y) + Kf(y))/y = -\frac{1}{2}\pi K^2$ so that $V^d(y)$ satisfies

$$\int_0^a \frac{V^d(t)}{y^2-t^2}\mathrm{d}t = -\tfrac{1}{2}\pi K^2, \qquad y \in (0,a). \tag{22}$$

There is a general formula for the inversion of integral equations of the type above with arbitrary right-hand sides and application of this for the particular right-hand side above gives

$$V^d(t) = \frac{D}{(a^2-t^2)^{1/2}} - K^2(a^2-t^2)^{1/2} \tag{23}$$

where D is a constant to be determined, whose origins can be traced back to transformation of the original integral equation (20) into (21). Thus, we substitute (23) back into (20) to determine D.

First we make use of an integral identity between $U^d(t)$ and $V^d(t)$, which is easily established from (21) and integration by parts, to obtain

$$\int_0^a L(k,t)U^d(t)\mathrm{d}t = k\int_0^a V^d(t)\cos kt dt = \tfrac{1}{2}\pi\left(kDJ_0(ka) - K^2aJ_1(ka)\right) \tag{24}$$

after using (23) and standard integral identities

$$\int_0^a \frac{\cos(ky)}{(a^2-y^2)^{1/2}}\mathrm{d}y = -\pi J_0(ka)/2 \qquad \text{and} \qquad \int_0^a (a^2-y^2)^{1/2}\cos(ky)\mathrm{d}y = \pi aJ_1(ka)/2k. \tag{25}$$

It follows from using (24) in (20) that

$$\widetilde{R}^d\mathrm{e}^{-Ky} + Ky - 1 = D\int_0^\infty \frac{J_0(ka)L(k,y)}{(k^2+K^2)}\mathrm{d}k - K^2a\int_0^\infty \frac{J_1(ka)L(k,y)}{k(k^2+K^2)}\mathrm{d}k, \qquad y\in(0,a). \tag{26}$$

With some effort, further integral relations can be established, in particular

$$\int_0^\infty \frac{J_0(ka)L(k,y)}{(k^2+K^2)}\mathrm{d}k = \mathrm{e}^{-Ky}K_0(Ka), \qquad \int_0^\infty \frac{J_1(ka)L(k,y)}{k(k^2+K^2)}\mathrm{d}k = \frac{(1-Ky)}{K^2a} - \frac{K_1(Ka)\mathrm{e}^{-Ky}}{K}. \tag{27}$$

Substituting these into (26) we find that the terms $1-Ky$ on each side of the equation cancel to leave

$$\widetilde{R}^d = DK_0(Ka) + KaK_1(Ka) \tag{28}$$

which determines D (in terms of $\widetilde{R}^d$). A relation for $\widetilde{R}^d$ follows from the first equation in (19) which can be written using the relation between U^d and V^d in (21) and integration by parts as

$$\widetilde{R}^d = -2\mathrm{i}\int_0^a U^d(y)\mathrm{e}^{-Ky}\mathrm{d}y = -2\mathrm{i}\int_0^a \cosh(Ky)V^d(y)\mathrm{d}y = -\mathrm{i}\pi\left(DI_0(Ka) - KaI_1(Ka)\right) \tag{29}$$

after susbtitution of (23) and using the results (25) with k replaced by iK.

Equations (28) and (29) may be combined to give

$$\widetilde{R}^d = \mathrm{i}\pi B\left(I_0K_1 + I_1K_0\right)Ka, \quad D = \mathrm{i}BC^{-1}Ka, \quad \text{where} \quad B^{-1} = K_0 + \mathrm{i}\pi I_0, \quad C^{-1} = \pi I_1 + \mathrm{i}K_1 \tag{30}$$

are defined in (9) and (10) and where the argument of the Bessel functions is Ka throughout. We are nearly in a position to determine R^u from (16) but first need to determine A^d from (14). We have that $R^d = BK_0(Ka)$ and it can also be shown, en route to the derivation of the Dean potential, that $\phi_x^d(0,y) \sim B/(a^2-y^2)^{1/2}$. It follows from the comments after (21) and from (23) that we require $A^dD + B = 0$ and so (16) becomes

$$R^u = 1 - R^d + BD^{-1}\widetilde{R}^d = B(\mathrm{i}\pi I_0 + D^{-1}\widetilde{R}^d) = B\pi\left(\mathrm{i}I_0 + C(I_0K_1 + I_1K_0)\right) = \pi I_1(Ka)C \tag{31}$$

using (30), which is the Ursell result.

To derive the Ursell potential from (11) we first use (10) and (17), (19) and (24) to show that

$$\phi^d(x,y) + A^d\psi^d(x,y) = -\mathrm{e}^{\mathrm{i}Kx-Ky} + \left(BK_0(Ka) + A^d\widetilde{R}^d\right)\mathrm{e}^{\mathrm{i}Kx-Ky} + K^2aA^d\int_0^\infty \frac{L(k,y)J_1(ka)\mathrm{e}^{-kx}}{k(k^2+K^2)}\mathrm{d}k$$

where the resulting integral involving $J_0(ka)$ vanishes since $B + A^dD = 0$. Also it can be shown that $BK_0(Ka) + A^d\widetilde{R}^d = \mathrm{i}CK_1(Ka)$ so that from (11)

$$\begin{aligned}
\phi^u(x,y) + \mathrm{e}^{\mathrm{i}Kx-Ky} &= \mathrm{i}K^{-1}\frac{\partial}{\partial x}\left(\phi^d(x,y) + A^d\psi^d(x,y)\right) \\
&= \mathrm{e}^{\mathrm{i}Kx-Ky} - \mathrm{i}CK_1(Ka)\mathrm{e}^{\mathrm{i}Kx-Ky} - \mathrm{i}KaA^d\int_0^\infty \frac{L(k,y)J_1(ka)\mathrm{e}^{-kx}}{(k^2+K^2)}\mathrm{d}k \\
&= C\left(\pi I_1(Ka)\mathrm{e}^{\mathrm{i}Kx-Ky} + \int_0^\infty \frac{L(k,y)J_1(ka)\mathrm{e}^{-kx}}{(k^2+K^2)}dk\right)
\end{aligned} \tag{32}$$

since $-\mathrm{i}KaA^d = C$. This is precisely the Ursell potential given by (9). We could equally have derived the connection potential $\psi^u(x,y)$ and used it in conjunction with the Ursell potential to derive the Dean potential.

The connection potentials described here are of academic interest only in the present context as both the Ursell and Dean solutions are well-known. However it may transpire in more complicated problems that one of the problems is more difficult than the other in which case the connection potentials would provide the link between them.

References

[1] Ursell, F. Proc. Camb. Phil. Soc. **42** (1947), 374.

[2] Dean, W.R. Proc. Camb. Phil. Soc. **41** (1945), 231.

[3] Havelock T.H. Phil. Mag. **8** (1929), 569.