Expansion formula for velocity potential for wave interaction with floating and submerged structures

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1 Introduction

In recent decades, there is a significant study on surface wave interaction with very large floating structures for ocean space utilization. An interesting aspect of these class of problems is to reduce the structural response of very large floating structures (VLFS). One of the approaches for mitigating structural response of a very large floating structure under wave action is with the help of a submerged flexible structures (see Wang et al. (2010)). Hassan et al. (2009) analyzed the surface wave interaction with submerged flexible plates of finite and semi-infinite length in two-dimensional as well as the three-dimensional problem involving a circular plate by the matching method. In the present paper, Fourier type expansion formula for the velocity potentials and associated orthogonal mode-coupling relations are derived in water of finite depth to deal with wave interaction with floating flexible structures in the presence of submerged flexible structures. The expansion formula are also derived in an alternate manner using Green's function technique.

2 Expansion formula

Under the assumption of the linearized theory of water waves and small amplitude structural response, the problem is considered in the two-dimensional Cartesian co-ordinate system with x-axis being in the horizontal direction and y-axis in the vertically downward positive direction. An infinitely extended thin elastic plate is floating at the mean free surface $y = 0$ in an infinitely extended fluid and another infinitely extended submerged flexible plate is kept horizontally at $y = h$ in the fluid domain as in Figure 1. Assuming that the fluid is inviscid, incompressible and irrotational and simple harmonic in time with angular frequency ω, the fluid motion is described by the velocity potentials $\Phi_j(x, y, t) = \text{Re}\{\phi_j(x, y)e^{-i\omega t}\}$ with subscript $j = 1$ referring to the fluid domain bounded by the floating and submerged plate and $j = 2$ referring to the fluid domain bounded by the submerged plate and bottom bed. Further, it is assumed that the deflection of the floating and submerged plates are of the forms $\zeta_j = \text{Re}\{\zeta_j(x)e^{-i\omega t)}\}$ with $j = 1$ refers to the floating plate and $j = 2$ refers to the submerged plate. The spatial velocity potential $\phi_j(x, y, t)$ satisfies the Laplace equation as given by

$$\nabla^2\phi_j = 0, \quad \text{in the respective fluid domain.} \tag{2.1}$$

The rigid bottom boundary conditions are given by

$$\frac{\partial \phi_2}{\partial y} = 0 \quad \text{at} \quad y = H. \tag{2.2}$$

The linearized kinematic condition on the submerged plate surface at $y = h$ as given by

$$\left.\frac{\partial \phi_2}{\partial y}\right|_{y=h+} = \left.\frac{\partial \phi_1}{\partial y}\right|_{y=h-}. \tag{2.3}$$

Assuming $m_{pi}\omega^2 << 1$ (as in Schulkes et al. (1987)), the mean free surface at $y = 0$, ϕ_1 satisfies

$$D_1 \frac{\partial^5 \phi_1}{\partial y^5} - N_1 \frac{\partial^3 \phi_1}{\partial y^3} + \frac{\partial \phi_1}{\partial y} + K\phi_1 = 0 \quad \text{on y} = 0,\ 0 < \text{x} < \infty. \tag{2.4}$$

On the submerged flexible plate at $y = h$, ϕ_1 and ϕ_2 satisfy

$$D_2 \frac{\partial^5 \phi_2}{\partial y^5} - N_2 \frac{\partial^3 \phi_2}{\partial y^3} + K(\phi_2 - \phi_1) = 0, \quad \text{for} \quad y = h,\ 0 < x < \infty, \tag{2.5}$$

where $D_i = E_i I_i/\rho g$, $N_i = Q_i/\rho g$ and $K = \omega^2/g$. In addition, assuming that a vertical

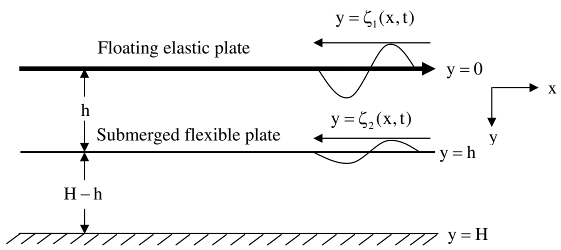

Figure 1: Schematic diagram of floating and submerged flexible plates in water of finite depth

wavemaker oscillates with frequency ω and amplitude $u(y)$ about its mean position on the wavemaker at $x = 0$, the spatial velocity potential ϕ satisfies

$$\frac{\partial \phi}{\partial x} = u(y) \quad \text{on} \quad x = 0, \tag{2.6}$$

for $0 < y < H$ except at $y = h$. Finally, the far field radiation condition is of the form given by

$$\phi(x, y) = \sum_{n=I}^{II} B_n g_n(y) e^{ip_n x} \quad \text{as} \quad x \to \infty, \tag{2.7}$$

where p_ns are the progressive flexural gravity wave modes generated due to the interaction of the surface gravity waves with the floating and the submerged flexible plates, $g_n(y)$ are the vertical eigenfunctions and B_n are associated with the unknown wave amplitude at far field.

2.1 Fourier type expansion formula

Using eigenfunction expansion method, the velocity potential $\phi(x, y)$ satisfying Eq.(2.1) along with the boundary conditions in Eqs.(2.2)-(2.5) in finite water depth is of the form

$$\phi(x, y) = \sum_{n=I,\ldots,X,1}^{\infty} B_n \psi_n(y) e^{ip_n x}, \quad \text{for} \quad x > 0, \tag{2.8}$$

$$\text{with } \psi_n(y) = \begin{cases} \{(D_2 p_n^4 - N_2 p_n^2) p_n \tanh p_n(H - h) - K\} \dfrac{L_1(ip_n, y)}{L_1(ip_n, h)}, & \text{for } 0 < y < h, \\ -K \cosh p_n(H - y)/\cosh p_n(H - h), & \text{for } h < y < H, \end{cases} \tag{2.9}$$

and B_ns are the unknowns to be determined with $L_1(ip_n, y) = iK\{p_n(D_1 p_n^4 - N_1 p_n^2 + 1)\cosh p_n y - K \sinh p_n y\}$. The eigenvalues $p_n, n = I, II, ..., IX, X$ in Eq.(2.8) in p satisfy the dispersion relation

$$\mathcal{G}(p) \equiv K - \frac{p(1 + D_1 p^4 - N_1 p^2)}{\mu} = 0, \tag{2.10}$$

with $\mu = \dfrac{K\{1+\coth ph \coth p(H-h)\} - (D_2p^4 - N_2p^2)p\coth ph}{K\{\coth ph + \coth p(H-h)\} - p(D_2p^4 - N_2p^2)}$. Keeping the realistic nature of the physical problem, it is assumed that the dispersion relation in Eq.(2.10) has two distinct positive real roots $p_n,\ n = I, II$, eight complex roots $p_n,\ n = III, ..., X$ of the form $a \pm ib$ and $-c \pm id$ and infinite number of purely imaginary roots $p_n,\ n = 1, 2, ...$ of the form $p_n = i\nu_n$ (which can be easily observed for specific problems through contour plots). The bounded characteristics of the far field behavior of the velocity potential in Eq.(2.7) yields $B_{VII} = ... = B_X = 0$ in Eq.(2.8). Further, it can be easily derived that the eigenfunctions ψ_ns satisfy the orthogonal mode-coupling relation given by

$$\begin{aligned}
\langle \psi_m, \psi_n \rangle &= \langle \psi_m, \psi_n \rangle_1 + \langle \psi_m, \psi_n \rangle_2 = E_n\delta_{mn} \quad \text{for all} \quad m = n = I, ..., VI, 1, 2, ... \\
\text{with } \langle \psi_m, \psi_n \rangle_1 &= \int_0^h F_m(y)F_n(y)dy - \frac{N_1}{K}\psi_m'(0)\psi_n'(0) + \frac{D_1}{K}\{\psi_m'(0)\psi_n'''(0) + \psi_m'''(0)\psi_n'(0)\}, \\
\langle \psi_m, \psi_n \rangle_2 &= \int_h^H \psi_m(y)\psi_n(y)dy - \frac{N_2}{K}\psi_m'(h)\psi_n'(h) + \frac{D_2}{K}\{\psi_m'(h)\psi_n'''(h) + \psi_m'''(h)\psi_n'(h)\}, \\
E_n &= \frac{-\mathcal{D}(ip_n, h)\sinh p_nh[\psi_n'(0)]^2\mathcal{G}'(p_n)}{2K^2p_n^2},
\end{aligned}$$

where $\mathcal{D}(ip_n, h) = -\{p_n(D_1p_n^4 - N_1p_n^2 + 1)\sinh p_nh - K\cosh p_nh\}$. The constants B_ns are given by

$$B_n = \frac{\langle u(y), \psi_n(y) \rangle}{ip_nE_n}. \tag{2.11}$$

Next, one of the important characteristics of the eigenfunctions $\psi_n(y)$s is mentioned without proof as a Theorem next.

Theorem. The eigenfunctions $\psi_n(y)$s in Eq.(2.9) are linearly dependent.

Proceeding in a similar manner as in Mondal and Sahoo (2012), the above theorem can be proved and details are deferred here.

2.2 Derivation of line source potentials

The symmetric wave source potential associated with surface gravity wave problems $G(x, y; x_0, y_0)$ (which is also referred as the Green's function) in the presence of floating and submerged elastic plates satisfies Laplace equation in the fluid region except at the structural boundaries and at the source point (x_0, y_0) along with the boundary conditions as in Eqs.(2.2)-(2.5). Near the source point (x_0, y_0), the Green's function behaves like

$$G \sim \frac{1}{2\pi}\ln(r) \quad \text{as} \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \to 0. \tag{2.12}$$

Assuming the symmetric property of the fundamental wave source potential about $x = x_0$, condition (2.12) yields (as in Mohapatra et al. (2011))

$$\frac{\partial G}{\partial x} = \delta(y - y_0)/2 \quad \text{on} \quad x = x_0. \tag{2.13}$$

Using the generalized identity

$$\int_0^\infty \delta(y-y_0)F(y)dy = \begin{cases} F(y_0) & \text{if} \quad y_0 > 0, \\ F(y_0)/2 & \text{if} \quad y_0 = 0, \end{cases} \tag{2.14}$$

and the expansion formula as in the previous Section, the source potential $G(x, y; x_0, y_0)$ is obtained as

$$G(x, y; x_0, y_0) = \sum_{n=I}^{VI} B_n\psi_n(y)e^{ip_n(x-x_0)} + \sum_{n=1}^{\infty} B_n\psi_n(y)e^{-\nu_n(x-x_0)} \quad \text{for} \quad x > x_0, \tag{2.15}$$

where B_ns are given by

$$B_n = \begin{cases} \dfrac{-\delta_1 \sinh p_n(H-h) L_1(ip_n; y_0)}{2p_n E_n K \mathcal{D}(ip_n, h)} & \text{for } 0 \leq y_0 < h, \\ \dfrac{-\sinh p_n(H-h) L_1(ip_n; h)}{2p_n E_n} - \dfrac{i \cosh p_n(H-h)}{2p_n E_n} & \text{for } y_0 = h, \\ \dfrac{-i\delta_1 \cosh p_n(H-y_0)}{2p_n E_n} & \text{for } h < y_0 \leq H, \end{cases}$$

and $p_n = i\nu_n$ for $n = 1, 2, 3, ...$ with $\delta_1 = 1$ for $y_0 \in (0, h) \cup (h, H)$ and $\delta_1 = 1/2$ for $y_0 = 0, H$ and p_n, ν_n, $\psi_n(y)$, $L_1(ip_n; y_0)$, E_ns, $\mathcal{D}(ip_n, h)$ being the same as in previous Section.

2.3 Expansion formula based on Green's function technique

In this subsection, using the source potential $G(x, y, x_0, y_0)$ derived in the previous subsection and Green's identity, the expansion formula for the flexural gravity wavemaker problem in the presence of a horizontal flexible plate is derived. In this case, the spatial velocity potential $\phi(x, y)$ satisfies the Laplace equation as in Eq.(2.1), along with the bottom boundary condition as in Eqs.(2.2), the boundary conditions on the floating and submerged flexible plates as in Eqs.(2.4) and (2.5). In order to derive an integral representation of the velocity potential in terms of the Green's function $G(x, y; x_0, y_0)$ satisfying the condition on the wavemaker as in Eq. (2.6), set

$$G^{mod}(x, y; x_0, y_0) = G(x, y; x_0, y_0) + G(-x, y; x_0, y_0), \tag{2.16}$$

with zero normal velocity on the wavemaker, i.e., $G^{mod}_x(0, y; x_0, y_0) = 0$. Using Green's identity, the velocity potential $\phi(x_0, y_0)$ is obtained as

$$\phi(x_0, y_0) = -\Big[2\int_{\Re} G(0, y; x_0, y_0)u(y)dy + \frac{2}{K}\Big[D_1\{G_{1yyy}\phi_{1xy} + G_{1y}\phi_{1xyyy}\} - N_1 G_{1y}\phi_{1xy}\Big]_{(x,y)=(0,0)} + \frac{2}{K}\Big[D_2\{G_{2yyy}\phi_{2xy} + G_{2y}\phi_{2xyyy}\} - N_2 G_{2y}\phi_{2xy}\Big]_{(x,y)=(0,h)}\Big]. \tag{2.17}$$

The Green's function and velocity potential derived here can be used to deal with gravity wave interaction with floating structure in the presence of submerged flexible structure of various configurations in finite water depth. Expansion formulae for the same class of problems can be derived in case of infinite water depth with suitable utilisation of mixed type of Fourier transform as in Mondal and Sahoo (2012) and alternately using Green's function technique as discussed in case of finite water depth.

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