Time dependent flexural gravity wavemaker problem

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1. Introduction

Currents with speeds exceeding 1m/s are observed in the near shore regions in many parts of the world and the mutual interaction between waves and the underlying currents alter the wave characteristics significantly. There is negligible progress in the literature to deal with wave structure interaction problems in the presence of current in time domain. Meylan et al. (2004) studied the wave interaction with a finite floating elastic plate in time domain using Laplace transform and Green's function technique. In the present paper, a class of initial boundary value problems associated with the time dependent flexural gravity wave maker problems are handled using Laplace transform method and Green's function technique in both the cases of finite and infinite water depths.

2. Mathematical formulation

In the present paper, the time dependent flexural gravity wavemaker problems are analyzed in two-dimensional cartesian co-ordinate system in both the cases of finite and infinite water depths assuming that a thin elastic plate of thickness d and density $\rho_i$is floating on the free surface of water. Assuming that the fluid is inviscid, incompressible of constant density ρ occupies the region $0 < x < \infty$, $0 < y < h$ in case of finite depth, ($0 < y < \infty$ in case of infinite depth). Further, it is assumed that the fluid motion is irrotational and there is a uniform current flowing with speed U along the direction of wave propagation. Thus, the total velocity potential $\oplus(x, y, t)$ is written as $\oplus(x, y, t) = Ux + \Phi(x, y, t)$ and the plate deflection is denoted as $\eta(x, t)$. Thus, the velocity potential $\Phi(x, y, t)$ satisfying the two dimensional Laplace equation is given by

$$\nabla^2\Phi = 0, \quad \text{in the fluid region,} \tag{1}$$

along with the bottom boundary condition

$$\partial_y\Phi = 0, \ \text{ on } y = h, \ \text{ and } \ \Phi, |\nabla\Phi| \to 0, \ \text{ as } \ y \to \infty, \tag{2}$$

in case of water of finite and infinite depths respectively. The linearized kinematic and dynamic conditions on the plate covered surface are given by

$$\partial_t\eta + U\partial_x\eta = \partial_y\Phi, \quad \text{on} \quad y = 0. \tag{3}$$

$$(D\partial_y^4 - Q\partial_y^2 + \gamma\partial_t^2 + g)\Phi_y = (\partial_t + U\partial_x)^2\Phi, \ \text{ on } \ y = 0, \tag{4}$$

where ∂ with suffix indicates the partial derivative with $D = EI/\rho$, $Q = N/\rho$, $\gamma = \rho_i d/\rho$, $I = d^3/(12(1-\nu^2))$, E is the Young's modulus, N is the compressive force, ν is the poison's ratio, and g is the acceleration due to gravity.

3. Green's function for flexural gravity waves

In this Section, the time dependent Green's functions $G(x, y; x_0, y_0, t)$ are derived for flexural gravity wave problems in case of finite and infinite water depths assuming that a point source of strength $m(t)$ is located at (x_0, y_0) in the fluid domain. The Green's function $G(x, y; x_0, y_0, t)$ satisfy the two dimensional Laplace equation as in Eq.(1) in the fluid domain except at (x_0, y_0), along with the boundary conditions as in Eqs. (2)–(4). In addition, the Green's function $G(x, y; x_0, y_0, t)$ satisfies the conditions

$$G \sim m(t)\ln r_1, \ \text{ as } \ r_1 \to 0, \quad \text{ and } \quad G, |\nabla G| \to 0, \quad \text{as} \quad r_1 \to \infty, \tag{5}$$

with $r_1 = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Further, the Green's function $G(x, y; x_0, y_0, t)$ satisfies the initial conditions

$$G, \partial_t G = 0, \quad \text{on } \ y = 0, \quad \text{at } \ t = 0. \tag{6}$$

In order to determine $G(x,y;x_0,y_0,t)$ associated with the aforementioned initial boundary value problem, the problem is transformed to a boundary value problem by using the Laplace transform in the time variable t. The transformed Green's function $\bar{G}(x,y;x_0,y_0,p)$ satisfies the two dimensional Laplace equation as in Eq. (1) in the fluid domain except at (x_0,y_0) along with the boundary conditions (2) and

$$(D\partial_y^5 - Q\partial_y^3 + g\partial_y^3 - p^2 - U^2\partial_x^2 - 2Up\partial_x)\bar{G} = 0 \ \text{ on } \ y = 0, \tag{7}$$

Finite water depth:
The transformed Green's function $\bar{G}(x,y;x_0,y_0,p)$ satisfying the governing Eq. (1) along with the boundary conditions (2) and (7) in case of finite water depth is given by

$$\bar{G} = \bar{m}(p)\Big[\ln\frac{r_1}{r_2} - \int_0^\infty \Big\{\frac{2e^{-kh}g_1(y,y_0)\cos k(x-x_0)}{k\cosh kh} + \sum_{m=I}^{II}\frac{\Omega_h^2 g_2(y,y_0)e^{i\epsilon_m k(x-x_0)}}{\{\Omega_h^2 + (p+i\epsilon_m Uk)^2\}g_3(k)}\Big\}dk\Big], \tag{8}$$

where $\Omega_h^2 = (Dk^4 - Qk^2 + g)k\tanh kh$, $r_2 = \sqrt{(x-x_0)^2 + (y+y_0)^2}$,

$$\left.\begin{aligned} g_1(y,y_0) &= \sinh ky \sinh ky_0, \quad g_3(k) = k\sinh kh\cosh kh, \\ g_2(y,y_0) &= \cosh k(h-y)\cosh k(h-y_0), \end{aligned}\right\}, \epsilon_m = \begin{cases} 1, & \text{for } m = I, \\ -1, & \text{for } m = II. \end{cases} \tag{9}$$

Using the inverse Laplace transform and convolution theorem, Eq. (8) yields

$$G(x,y,x_0,y_0,t) = m(t)\Big\{\ln\frac{r_1}{r_2} - 2\int_0^\infty \frac{e^{-kh}g_1(y,y_0)\cos k(x-x_0)}{k\cosh kh}dk\Big\}.$$
$$- 2\int_0^\infty \frac{\Omega_h g_2(y,y_0)}{g_3(k)}\Big\{\int_0^t m(\tau)\sin\Omega_h(t-\tau)\cos k\{(x-x_0) - U(t-\tau)\}d\tau\Big\}dk. \tag{10}$$

Assuming that the motion is simple harmonic in time with angular frequency ω, the associated Green's function is written in the form $G(x,y;x_0,y_0,t) = Re[\mathcal{G}(x,y;x_0,y_0)e^{-i\omega t}]$. Thus, the spatial Green's function $\mathcal{G}(x,y;x_0,y_0)$ satisfy Eq. (1) along with the boundary conditions (2) and (7). Hence, substituting $p^2 = -\omega^2$ and $\bar{m}(p) = 1/(2\pi)$, Eq. (8) yields

$$\mathcal{G} = \frac{1}{2\pi}\Big[\ln\frac{r_1}{r_2} - \int_0^\infty \Big\{\frac{2e^{-kh}g_1(y,y_0)\cos k(x-x_0)}{k\cosh kh} + \sum_{m=I}^{II}\frac{\Omega_h^2 g_2(y,y_0)e^{i\epsilon_m k(x-x_0)}}{\{\Omega_h^2 - (\omega+\epsilon_m Uk)^2\}g_3(k)}dk\Big\}\Big]. \tag{11}$$

Now, applying Cauchy residue theorem, $\mathcal{G}(x,y;x_0,y_0)$ is rewritten as

$$\mathcal{G} = \sum_{m=I}^{II}\Big[\sum_{n=0,I}^{II}\frac{\delta_{m,n}M_1(k_n)f_n(y)f_n(y_0)}{L'(k_n,\epsilon_m)}e^{i\epsilon_m k_n(x-x_0)} + \sum_{n=1}^{\infty}\frac{M_1(k_n)f_n(y)f_n(y_0)}{L'(k_n,\epsilon_m)}e^{-p_n|x-x_0|}\Big], \tag{12}$$

where $M_1(k_n) = -i\kappa_n(Dk_n^4 - Qk_n^2 + g)$, $f_n(y) = \cosh k_n(h-y)/\cosh k_n h$, with $L(k,\epsilon_m) = \Omega_h^2 - (\omega + \epsilon_m Uk)^2$, $\delta_{I,0} = \delta_{II,0} = \delta_{I,I} = \delta_{II,II} = 1$, $\delta_{I,II} = \delta_{II,I} = 0$, $\kappa_n = 1/2$ for $n = 0$ and one otherwise. The expansion formula in Eq. (12) reduces to the formula by Manam et al. (2006) in the absence of current.

Infinite water depth:
Proceeding in a similar manner as in case of finite depth, the transformed Green's function $\bar{G}(x,y;x_0,y_0,p)$ in case of infinite water depth is obtained as

$$\bar{G}(x,y;x_0,y_0,p) = \bar{m}(p)\Big\{\ln\frac{r_1}{r_2} - \sum_{m=I}^{II}\int_0^\infty \frac{\Omega^2 e^{i\epsilon_m k(x-x_0) - k(y+y_0)}}{k\{\Omega^2 + (p+i\epsilon_m Uk)^2\}}dk\Big\}. \tag{13}$$

where $\Omega^2 = (Dk^4 - Qk^2 + g)k$. Using inverse Laplace transform and convolution theorem from Eq. (13), the time dependent Green's function $G(x,y;x_0,y_0,t)$ is obtained as

$$G = m(t)\ln\frac{r_1}{r_2} - \int_0^\infty\int_0^t \frac{2\Omega e^{-k(y+y_0)}}{k}m(\tau)\sin\Omega(t-\tau)\cos k\{x - x_0 - U(t-\tau)\}d\tau dk. \tag{14}$$

In case of simple harmonic motion proceeding in a similar manner as in case of finite water depth the spatial Green's function $\mathcal{G}(x,y;x_0,y_0)$ is obtained as

$$\mathcal{G}(x,y;x_0,y_0)=\frac{1}{2\pi}\Big[\ln\frac{r_1}{r_2}+\int_0^\infty\frac{e^{-ky}}{2}\Big\{C^+(k)e^{ik(x-x_0)}+C^-(k)e^{-ik(x-x_0)}\Big\}dk\Big], \tag{15}$$

with $C^\pm=-\Omega^2e^{-ky_0}/[k\{\Omega^2-(\omega\pm Uk)^2\}]$. Applying Cauchy residue theorem, the Green's function $\mathcal{G}(x,y;x_0,y_0)$ can be rewritten as

$$\mathcal{G}=\sum_{m=I}^{II}\sum_{n=0,I}^{II}\frac{\delta_{m,n}M_1(k_n)e^{-k_n(y+y_0)}}{F_1(k_n,\epsilon_m)}e^{i\epsilon_mk_n(x-x_0)}-\int_0^\infty\frac{M(k,y)M(k,y_0)}{k\triangle(k)}e^{-k(x-x_0)}dk, \tag{16}$$

where $F_1(k,\epsilon_m)=5Dk^4-3Qk^2+g-2U(\omega+U\epsilon_mk)$, $M(k,y)=\Omega^2k\cos ky-(\omega+iUk)^2\sin ky$ and $\Delta(k)=\Omega^4k^2+(\omega+iUk)^4$. It can be easily proved that the series as in Eq.(12) and integral in Eq.(15) are absolutely convergent.

4. Flexural gravity wave-maker problem

In this Section, the Green's function derived in the aforementioned subsection will be used to find the expansion formulae for the velocity potentials associated with the flexural gravity wave maker problems in time domain in the presence of current in both the cases of finite and infinite water depths. Here, the velocity potential $\Phi(x,y,t)$ satisfies the governing Eq. (1), the initial conditions (6) along with the boundary conditions in Eqs. (2)–(4). Assuming that a wave maker located at $x=0$ is oscillating with velocity $U_1(y,t)$. Thus, the boundary condition on the wave maker is given by

$$\frac{\partial\Phi}{\partial x}=U_1(y,t)+U,\quad\text{on}\quad x=0. \tag{17}$$

In order to find the velocity potential $\Phi(x,y,t)$, the initial value problem is converted to a boundary value problem in $\bar\Phi(x,y,p)$ where $\bar\Phi(x,y,p)$ is the Laplace transform of $\Phi(x,y,t)$. Then, Green's identity is applied to the boundary value problem associated with the transformed functions $\bar\Phi(x,y,p)$ and a suitable chosen Green's function $\bar G^{mod}(x,y;x_0,y_0,p)$ defined by

$$\bar G^{mod}(x,y;x_0,y_0,p)=\bar G(x,y;x_0,y_0,p)+\bar G(x,y;-x_0,y_0,p), \tag{18}$$

where $\bar G(x,y;x_0,y_0,p)$ is the transformed Green's function as in Eqs. (8) and (13) in cases of finite and infinite water depths respectively. From Eq. (18), it is clear that $\bar G^{mod}_x(x,y;x_0,y_0,p)=0$. Applying Green's identity to the boundary value problem in $\bar\Phi(x,y,p)$ and $\bar G^{mod}(x,y;x_0,y_0,p)$ and proceeding in a similar manner as in Manam et al. (2006), the velocity potential $\bar\Phi(x_0,y_0,p)$ is obtained as

$$\bar\Phi(x_0,y_0,p)=A(p)-2\int_R\Big\{\bar U_1(y,p)+(U/p)\Big\}\bar G(0,y;x_0,y_0,p)dy, \tag{19}$$

$$A(p)=\int_0^\infty\Big[\frac{2U^2}{p^2}\Big(\bar\Phi_{xx}\bar G_y-\bar G_{xx}\bar\Phi_y\Big)-\frac{4U}{p}\Big(\bar G_x\bar\Phi_y-\bar\Phi_x\bar G_y\Big)\Big]_{y=0}dx$$
$$-\frac{2}{p^2}\Big\{D\Big(\bar\Phi_{yyyx}\bar G_y+\bar G_{yyy}\bar\Phi_{xy}\Big)-Q\bar\Phi_{xy}\bar G_y\Big\}_{(x,y)=(0,0)}, \tag{20}$$

where R varies from 0 to ∞ and 0 to h in the case of infinite and finite water depth respectively. Taking the inverse Laplace transform of Eq. (19) and using convolution theorem the velocity potential $\Phi(x_0,y_0,t)$ in case of finite water depth is obtained as

$$\Phi(x_0,y_0,t)=L^{-1}\{A(p)\}-2\int_0^h\int_0^t\{U_1(y,t-\tau)+U\}\Big[m(\tau)\Big\{\ln\frac{r_1}{r_2}$$
$$-2\int_0^\infty\Big(\frac{e^{-kh}g_1(y,y_0)\cos kx_0}{k\cosh kh}-\frac{\Omega_hg_2(y,y_0)F_h(\tau,x,x_0)}{g_3(k)}\Big)dk\Big\}\Big]d\tau dy. \tag{21}$$

Similarly, taking inverse Laplace transform of Eq. (19) and using convolution theorem the velocity potential $\Phi(x_0, y_0, t)$ in case of infinite water depth is obtained as

$$\Phi(x_0,y_0,t) = L^{-1}[A(p)] - 2\int_0^\infty \int_0^t \{U_1(y,t-\tau)+U\}\Big\{m(\tau)\ln\frac{r_1}{r_2} - 2\int_0^\infty \Omega e^{-k(y+y_0)}F(\tau,x,x_0)dk\Big\}d\tau dy, \tag{22}$$

where

$$F_h(t,x,x_0) = \int_0^t m(\tau)\sin\Omega_h(t-\tau)\cos k[x-x_0-U(t-\tau)]d\tau, \tag{23}$$

$$F(t,x,x_0) = \int_0^t m(\tau)\sin\Omega(t-\tau)\cos k[x-x_0-U(t-\tau)]d\tau. \tag{24}$$

In particular, for $p^2 = -\omega^2$ and $\bar{m}(p) = 1$, from Eqs. (8) and (13), the spatial velocity potential $\phi(x, y)$ associated with the time harmonic motion with angular frequency ω are derived as

$$\phi(x,y) = \sum_{m=I}^{II}\Big\{\sum_{n=0,I}^{II} A_{m,n}(k_n)f_n(y)e^{i\epsilon_m k_n x} + \sum_{n=1}^{\infty} B_{m,n}(k_n)f_n(y)e^{-k_n x}\Big\}, \text{ for finite depth,} \tag{25}$$

$$\phi(x,y) = \sum_{m=I}^{II}\sum_{n=0,I}^{II} A_{1m,n}(k_n)e^{-k_n y}(y)e^{i\epsilon_m k_n x} + \int_0^\infty \frac{\hat{T}(k)M(k,y)e^{-kx}dk}{\triangle(k)}, \text{ for infinite depth,} \tag{26}$$

where

$$\left.\begin{aligned}
A_{m,n}(k_n) &= \frac{2\delta_{m,n}M_1(k_n)}{L'(k_n,\epsilon_m)}\Big[\int_0^h \{U_1(y)+U/(i\omega)\}f_n(y)dy \\
&\quad - \int_0^\infty \Big\{\alpha(k_n)+\beta(k_n)\Big\}e^{i\epsilon_m k_n x}dx + \frac{k_n\tanh k_n h}{\omega^2}\{Q\beta_1 - Dk_n^2\beta_1 - D\beta_2\}\Big], \\
A_{1m,n}(k_n) &= \frac{\delta_{m,n}M_1(k_n)}{F(k_n,\epsilon_m)}\Big[\int_0^\infty \{U_1(y)+U/(i\omega)\}e^{-k_n y}dy \\
&\quad - \int_0^\infty \Big\{\alpha_1(k_n)+\beta_1(k_n)\Big\}e^{i\epsilon_m k_n x}dx + \frac{k_n}{\omega^2}\{Q\beta_1 - Dk_n^2\beta_1 - D\beta_2\}\Big], \\
\hat{T}(k) &= \Big[\frac{1}{k}\int_0^\infty \{U_1(y)+U/(i\omega)\}M(k,y)dy + \frac{(\omega+Uik)^2}{\omega^2}\{(Q-Dk^2)\beta_1 + D\beta_2\}\Big] \\
&\quad + \frac{2iU}{\omega^2}\int_0^\infty \Big\{(\omega+Uk)^2(iU\beta_4 - 2\omega\beta_5) - \Omega^2 k\beta_3(2\omega+iUk)\Big\}e^{-kx}dx,
\end{aligned}\right\} \tag{27}$$

with $B_{m,n}(k_n) = A_{m,n}(ik_n)$, $\beta_1 = \phi_{xy}(0,0)$, $\beta_2 = \phi_{yyyx}(0,0)$, $\beta_3 = \phi_y(x,0)$, $\beta_4 = \phi_{xx}(x,0)$, $\beta_5 = \phi_x(x,0)$, $\beta(k) = \frac{U^2k}{\omega^2}\{k\beta_3 - \beta_4\tanh kh\}$, $\alpha(k) = -\frac{2iUk}{\omega}\{i\epsilon_m\beta_3 + \beta_5\tanh kh\}$, $\alpha_1(k_n)$ and $\beta_1(k_n)$ can be obtained by taking $h \to \infty$. It is easy to check that in the absence of current the expansion formula for time harmonic velocity potential as in Eqs. (25) and (26) is same as in Manam et al. (2006). Particular cases with computational results associated with the initial value wavemaker problem will be presented in the workshop.

References

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