Near Trapping and the Singularity Expansion Method

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1 Introduction

Trapped modes are very special solutions to the linear water wave problem in which a solution with finite energy but no decay in time decay exists (McIver, 1996). A near trapped mode is a wave which has a very slow decay in time (and correspondingly slow growth in distance) and they may be thought of as corresponding to slight perturbations of a trapping structure geometry, although they occur in many different situations (Evans & Porter, 1997; Meylan & Eatock Taylor, 2009) and are a feature of multiple scattering and resonance.

It is well known that strong connections exist between the frequency and time domain solutions and this connection is exploited in the Cummins method (Cummins, 1962), which is the standard solution method in the time-domain for determining body motions (it cannot be used to determine the fluid motion). However, a more direct connection between the frequency and time domain solution can be derived using the generalized eigenfunction method (Fitzgerald & Meylan, 2011). The singularity expansion method is a method for approximating time dependent wave problems by a deformation of the contour of integration which appears in the inverse Laplace transform (Hazard & Loret, 2007).

2 Mathematical Formulation

Positions are described in Cartesian coordinates $\boldsymbol{x} = (x, z)$ with z being directed vertically upwards. The fluid is two-dimensional with domain Ω of constant finite depth h_{-1} for $x < -l$ and h_1 for $x > l$. The sea bed is positioned at $z = -h(x)$, the free-surface is at $z = 0$, and the domain extends to infinity in the horizontal directions. The linearized boundary conditions can be adopted. The time-dependent motions are described by the velocity potential $\Phi(\boldsymbol{x}, t)$ which satisfies

$$\Delta\Phi(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1a}$$

$$\partial_n\Phi = 0, \quad \boldsymbol{x} \in \partial\Omega_B, \tag{1b}$$

$$\partial_n\Phi = 0, \quad z = -h(x), \tag{1c}$$

where ∂_n is the outward normal, $\partial\Omega_B$ is the wetted body surface and $z = h(x)$ is the sea floor. On the free-surface $\partial\Omega_F$

$$\partial_z\Phi = \partial_t\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{1d}$$

$$\partial_t\Phi = -\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{1e}$$

where ζ is the free surface displacement. These equations have been non-dimensionalized. An initial disturbance on the free-surface is given by

$$\zeta(\boldsymbol{x}, 0) = \zeta_0(x)\big|_{t=0}, \quad \boldsymbol{x} \in \partial\Omega_F. \tag{2}$$

We assume that $\Phi(\boldsymbol{x}, 0) = 0$. A finite energy condition must also be satisfied.

2.1 Frequency-domain solution

Given that the motions are assumed harmonic for all time we can write $\Phi = \mathrm{Re}\{\phi \mathrm{e}^{-i\omega t}\}$ and $\zeta = \mathrm{Re}\{\xi e^{-i\omega t}\}$ so that equations (1a-1e) become

$$\Delta\phi = 0, \quad \boldsymbol{x} \in \Omega, \tag{3a}$$

$$\partial_n\phi = 0, \quad \boldsymbol{x} \in \partial\Omega_B, \tag{3b}$$

$$\partial_n\phi = 0, \quad z = -h(x), \tag{3c}$$

$$-\mathrm{i}\omega\xi = \partial_z\phi, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{3d}$$

$$\mathrm{i}\omega\phi = \xi, \quad \boldsymbol{x} \in \partial\Omega_F. \tag{3e}$$

In the frequency domain, the initial conditions given by equations (2) are replaced by conditions at infinity. We assume a wave of the form

$$\phi_\kappa^I(\boldsymbol{x}, k_\kappa) = \mathrm{e}^{-\mathrm{i}\kappa k_\kappa x}\frac{\cosh k_\kappa(z - h_\kappa)}{\cosh k_\kappa h_\kappa}, \tag{4}$$

is incident from negative infinity ($\kappa = -1$) or incident from positive infinite ($\kappa = 1$) where k_κ is the wavenumber given by the positive real solution to the dispersion equation $\omega^2 = k_\kappa \tanh k_\kappa h_\kappa$. We decompose the total potential as the sum of an incident and scattered wave potential

$$\phi_\kappa(\boldsymbol{x}, k_\kappa) = \phi_\kappa^I(\boldsymbol{x}, k_\kappa) + \phi_\kappa^S(\boldsymbol{x}, k_\kappa), \tag{5}$$

where the scattering potential ϕ_κ^S must satisfy the radiation condition .

3 Solutions of the time-dependent problem.

We define the Dirichlet to Neumann operator $\partial_n \boldsymbol{G}$ by

$$\partial_n \boldsymbol{G}\psi = \partial_n \Phi|_{z=0}, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{6}$$

where Φ is the solution to

$$\Delta\Phi = 0, \quad \boldsymbol{x} \in \Omega, \tag{7a}$$

$$\partial_n \Phi = 0, \quad \boldsymbol{x} \in \partial\Omega_B, \tag{7b}$$

$$\partial_n \Phi = 0, \quad z = -h(x), \tag{7c}$$

$$\Phi = \psi, \quad z = 0, \, \boldsymbol{x} \in \partial\Omega_F. \tag{7d}$$

Therefore, equations (1a-1e) can be written as

$$\partial_t^2 \zeta + \partial_n \boldsymbol{G}\zeta = 0. \tag{8}$$

3.1 Generalized eigenfunction expansion method

The generalized eigenfunction method (GEM) presented here is a slight modification of that presented in Meylan (2009) to allow for non-constant depth and to include the possibility of a trapped mode. The evolution operator $\partial_n \boldsymbol{G}$ is self-adjoint in the Hilbert space given by the following inner product

$$\langle \zeta, \eta \rangle_{\mathcal{H}} = \int_{\partial\Omega_F} \zeta \eta^* \, \mathrm{d}x, \tag{9}$$

where * denotes complex conjugate. The eigenfunctions of $\partial_n \boldsymbol{G}$ satisfy

$$\partial_n \boldsymbol{G}\psi = \omega^2 \psi. \tag{10}$$

Equation (10) is nothing more than equations (3). We define the eigenfunctions by restricting the frequency domain solution for an incident wave to the free surface, i.e.,

$$\psi_\kappa(x, k) = \phi_\kappa(\boldsymbol{x}, k_\kappa)|_{\boldsymbol{x} \in \Omega_F}. \tag{11}$$

It is possible for there to exist point spectra for this operator which correspond to the existence of a trapped mode. In this case

$$\partial_n \boldsymbol{G}\psi_p = \omega_p^2 \psi, \tag{12}$$

but

$$\langle \psi_p, \psi_p \rangle_{\mathcal{H}} < \infty. \tag{13}$$

The GEM allows us to calculate the time-dependent solutions using these eigenfunctions as follows

$$\begin{aligned} \zeta(x,t) = \mathrm{Re}\Big\{ \int_0^\infty \sum_{\kappa \in \{-1,1\}} \langle \zeta_0(x), \psi_\kappa(x,k) \rangle_{\mathcal{H}} \\ \times \psi_\kappa(x,k) \left. \frac{\mathrm{d}k}{\mathrm{d}\omega} \right|_{h=h_\kappa} e^{-\mathrm{i}\omega t} \, \mathrm{d}\omega \Big\} \\ + \mathrm{Re}\left\{ \sum_{p \in \Lambda} \frac{\langle \zeta_0(x), \psi_p(x) \rangle_{\mathcal{H}}}{\langle \psi_p(x), \psi_p(x) \rangle_{\mathcal{H}}} \psi_p(x) \mathrm{e}^{-\mathrm{i}\omega_p t} \right\}, \end{aligned} \tag{14}$$

where Λ is the set of trapped mode points (which is the empty set in the case no trapped waves are present).

3.2 Fourier/Laplace transform solution of time-domain equations

The Fourier/Laplace transform and its inverse is given by

$$\hat{f}(\sigma) = \int_0^\infty f(t) e^{i\sigma t} \, \mathrm{d}t, \; f(t) = \frac{1}{2\pi} \int\!\!\!\!\!\!- _{-\infty}^{\infty} \hat{f}(\sigma) e^{-i\sigma t} \, d\sigma,$$

where the integration is taken above any poles on the real axis (which correspond to trapped modes in our case). The Fourier/Laplace transform of equation (8) and the initial condition (2) gives

$$-\sigma^2 \hat{\zeta} + \partial_n \boldsymbol{G}\hat{\zeta} = -\mathrm{i}\sigma\zeta_0. \tag{15}$$

The solution for the displacement is given by

$$\begin{aligned} \zeta(x,t) &= \frac{1}{2\pi} \int\!\!\!\!\!\!-_{-\infty}^{\infty} - \left(\partial_n \boldsymbol{G} - \sigma^2 \right)^{-1} \mathrm{i}\sigma\zeta_0 \mathrm{e}^{-\mathrm{i}\sigma t} \, \mathrm{d}\sigma \\ &= \mathrm{Re}\left\{ \frac{1}{\pi} \int\!\!\!\!\!\!-_0^\infty - \left(\partial_n \boldsymbol{G} - \sigma^2 \right)^{-1} \mathrm{i}\sigma\zeta_0 \mathrm{e}^{-\mathrm{i}\sigma t} \, \mathrm{d}\sigma \right\}. \end{aligned} \tag{16}$$

3.3 Singularity expansion method

The singularity expansion method (SEM) is based on deforming the contour of integration and writing the integral (16) as a sum over the poles and ignoring the contribution from branch cuts or the integral at infinity. The poles are the solutions to

$$\left(\partial_n \boldsymbol{G} - \sigma^2\right) \chi_p = 0, \tag{17}$$

where we do not restrict χ_p to have finite energy and we consider complex σ. We also define the mode associated with the adjoint operator, $\bar{\chi}_p$, as

$$\left(\partial_n \boldsymbol{G} - \sigma^2\right)^* \bar{\chi}_p = 0, \tag{18}$$

where the star denotes the adjoint operator. If we approximate the solution to equation (16) by the contribution from the only the poles we obtain

$$\zeta(x,t) \approx \operatorname{Re}\left\{ \sum_p \frac{\langle -2\sigma_p \zeta_0(x), \bar{\chi}_p(x) \rangle_{\mathcal{H}}}{\langle (\partial_n \boldsymbol{G} - \sigma^2)' \chi_p, \bar{\chi}_p \rangle_{\mathcal{H}}} \chi_p \mathrm{e}^{-\mathrm{i}\omega_p t} \right\}. \tag{19}$$

4 Numerical solution

We describe here a numerical method which allows us to solve for the the Fourier/Laplace solution even for complex frequency values and which we can use to calculate the SEM solution. We use a boundary element method combined with a matched vertical eigenfunction expansion. A finite domain $\bar{\Omega}$ is defined by restricting Ω to $|x| < l$. We discretize the boundary of Ω with a set of constant panels. Outside of the domain $\bar{\Omega}$, Ω consists of two semi-infinite domains containing no bodies where the fluid depth is constant where the solution can be found by an eigenfunction expansion. The method is described in detail in Wang & Meylan (2002) and is similar to the modified finite element method used in Hazard & Lenoir (1993).

The constant panel boundary element method gives the matrix equation

$$\frac{1}{2}\boldsymbol{\phi} = \boldsymbol{H_n}\boldsymbol{\phi} - \boldsymbol{H}\boldsymbol{\phi_n}, \tag{20}$$

where $\boldsymbol{\phi}$ ($\boldsymbol{\phi_n}$) is the vector of potential (derivative) values on the boundary panels and $\boldsymbol{H}$ ($\boldsymbol{H_n}$) is the matrix corresponding to the Green function (normal derivative of the Green function).

To solve equation (20) we need to find a relationship between the normal derivative and the potential on the boundary. The boundary is divided into four regions. $\partial\Omega_1$ is the body and sea floor, $\partial\Omega_2$ is the free surface, $\partial\Omega_3$ is the left hand vertical boundary, and $\partial\Omega_4$ is the right hand vertical boundary.

The boundary condition on $\partial\Omega_3$ and $\partial\Omega_4$ is the most complicated as it depends non-trivially on the frequency. We use an integral relation to expresses the normal outward derivative in terms of an expansion in the vertical eigenfunctions,

$$\partial_n \phi = \boldsymbol{Q_{-1}}\phi, \quad \text{on} \quad \partial\Omega_3. \tag{21}$$

where $\boldsymbol{Q_{-1}}$ is given in Wang & Meylan (2002). We apply a similar derivation on the boundary $\partial\Omega_4$. The normal derivatives can be expressed in terms of the potential on all four boundaries,

$$\partial_n \boldsymbol{\phi} = 0, \quad \text{on} \quad \partial\Omega_1, \tag{22a}$$
$$\partial_n \boldsymbol{\phi} = \sigma^2 \boldsymbol{\phi}, \quad \text{on} \quad \partial\Omega_2, \tag{22b}$$
$$\partial_n \boldsymbol{\phi} = \boldsymbol{Q_{-1}}\phi, \quad \text{on} \quad \partial\Omega_3, \tag{22c}$$
$$\partial_n \boldsymbol{\phi} = \boldsymbol{Q_1}\phi, \quad \text{on} \quad \partial\Omega_4. \tag{22d}$$

We write this condition as

$$\boldsymbol{\phi}_n = \boldsymbol{A}(\sigma)\boldsymbol{\phi}, \tag{23}$$

where we explicitly included the dependence of $\boldsymbol{A}$ on the parameter σ.

For the case of the Fourier/Laplace transform solution we have

$$\left(\frac{1}{2} - \boldsymbol{H_n} + \boldsymbol{H}\boldsymbol{A}\right)\boldsymbol{\phi} = \boldsymbol{H}\boldsymbol{f}_0, \tag{24}$$

where $\boldsymbol{f}_0 = 0$ on $\partial\Omega_i$ except

$$\boldsymbol{f}_0 = \zeta_0, \quad \text{on} \quad \partial\Omega_2. \tag{25}$$

We locate the poles and vector $\boldsymbol{\chi}_p$ by searching for the values σ_p for which

$$\left(\frac{1}{2} - \boldsymbol{H_n}\boldsymbol{\phi} + \boldsymbol{H}\boldsymbol{A}(\sigma_p)\right)\boldsymbol{\chi}_p = 0, \tag{26}$$

and we define the adjoint vector $\bar{\boldsymbol{\chi}}_p$ by

$$\left(\frac{1}{2} - \boldsymbol{H_n} + \boldsymbol{H}\boldsymbol{A}(\sigma_p)\right)^* \bar{\boldsymbol{\chi}}_p. \tag{27}$$

We can then express the SEM numerically using our matrix approximation of the operator as

$$\zeta(x,t) \approx \operatorname{Re}\left\{ \sum_p \frac{\langle\, 2\sigma_p \boldsymbol{f}_0, \bar{\boldsymbol{\chi}}_p(x) \rangle \chi_p(x)}{\langle \left\{\boldsymbol{H}\left(\frac{1}{2} - \boldsymbol{H_n} + \boldsymbol{H}\boldsymbol{A}\right)\right\}' \boldsymbol{\chi}_p, \bar{\boldsymbol{\chi}}_p \rangle} \mathrm{e}^{-\mathrm{i}\omega_p t} \right\}, \tag{28}$$

where the inner product is the standard inner product for vectors. Note that by $\boldsymbol{\chi}_p$ we mean the vector defined on the boundary of $\bar{\Omega}$ while χ_p is the value restricted to the surface, and defined outside the region $\bar{\Omega}$ if needed. We require the region $|x| < l$ to enclose the initial condition. The derivative of the matrix is calculated numerically.

5 Results

We consider an initial surface displacement of the form

$$\zeta_0(x) = \mathrm{e}^{-20(x-0.2)^2}. \tag{29}$$

The exact solution is given by the solid line and is calculated using the GEM (equation (14)). The SEM solution (calculated using equation (28)) is given by the dashed line.

Figure 1 shows the exact and SEM solution for the case of two semi-circular fixed bodies with radius 0.1 centered at $(\pm 1, 0)$. The fluid depth is constant $h = 1$ and is not plotted but the circular bodies are plotted for illustration. Sixteen poles were used in the SEM approximation.

6 Summary

We have shown that the phenomena of trapping and near trapping in the time domain can be connected via the formula derived using the SEM. We have derived a practical method to implement the SEM and implemented it numerically.

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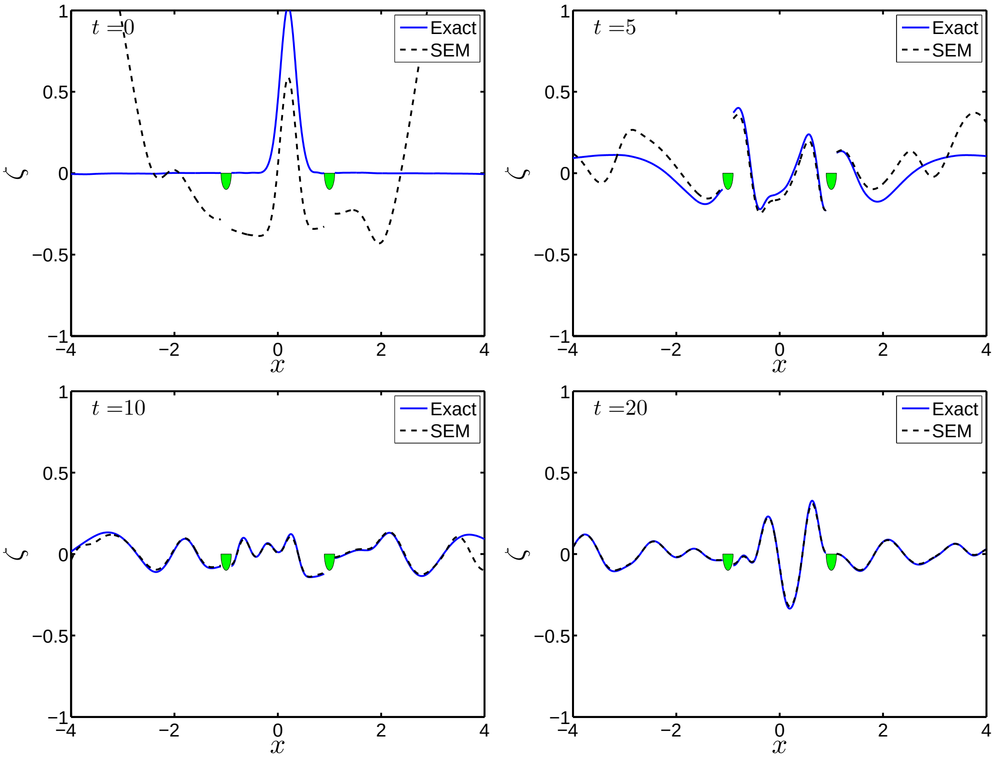

Figure 1: The exact (solid line) and SEM (dashed line) solution for the times shown. The scattering bodies are plotted for illustration.