Shapes of Zero Wave Resistance and Trapped Modes

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1. Introduction

The energy reduction at the transportation on the water is an actual problem. The searching for vessel shapes with minimal and zero wave resistance is an important part of them. It is possible to search of shapes of zero wave resistance at least by two ways.

The first is a modification of the body shape so that the wave resistance is minimal. The other method is based on a bijection of shape and pressure distribution that is generated. It is the searching of pressure distribution which gives the optimal shape. The first method can be called a direct, the second - an inverse. It is possible to search for optimal shape by combination of these methods. The shapes of zero wave resistance are considered by inverse method for several types of pressure distribution in [1]. In the present work the inverse method on a base of generalized functions technique is developed and some theoretical aspects of this approach are examined. The relation with trapped modes is considered.

2. Conditions for existence of the shapes with zero wave resistance

We consider the motion of a body with constant speed V_0 on a surface of ideal and incompressible liquid in assumptions of the linear theory of waves. The coordinate system is associated to the body and the x axis is directed against the velocity of the inverse flow. The fluid motion is irrotational and the potential $-V_0 x+\varphi(X,y,t)$ exists, where φ is the potential of disturbance velocity, $X=x\in R^1$ in 2D-theory and $X=(x,z)\in R^2$ in 3D-theory, y is the vertical axis, t is a time. The free surface shape is described by the function $y=\eta(X,t)$, $X\in R^n\setminus S(t)$, $n=1,2$, $S(t)$ is the projection on R^n of a point set of free surface and body surface intersection..We consider the shapes, which we can describe by an unique function $y=f(X,t)$, $X\in S(t)$ to simplify the analysis.

The problem is formulated as unsteady in order to avoid the analysis of diffraction conditions on infinity. The required steady solution with correct conditions on infinity is easier obtained by limiting process at $t\to\infty$. In addition, at $V_0=0$ it is the problem about a floating body on a wave surface. In this case, the steady problem will be the problem of steady oscillations under wave action and a searching for body shape which does not make resistance to a breaking wave.

The boundary conditions on the border of fluid and solid body $\Gamma(t)$ include the Bernoulli integral

$$N\varphi+(\nabla\varphi)^2/2=-gf(X,t)-\overline{p}(X,y,t), \qquad (1)$$

and kinematic condition

$$\nabla\varphi\cdot\nabla(y-f(X,t))=Nf(X,t),\ X\in S(t), \qquad (2)$$

where $N=\dfrac{\partial}{\partial t}-V_0\dfrac{\partial}{\partial x}$, $\quad \overline{p}(X,y,t)=\dfrac{p_a(X,y,t)-p_0}{\rho}$

is a relative pressure in the fluid, p_0 is an absolute pressure on the free surface, ρ is the density of fluid. The conditions on a free surface are the same but linearized and with zero pressure:

$$N\varphi=-g\eta(X,t),\quad y=-0,\ X\in R^n\setminus S(t), \quad (3)$$

$$\frac{\partial\varphi}{\partial y}=N\eta(X,t),\quad y=-0,\ X\in R^n\setminus S(t), \quad (4)$$

If the depth of fluid is h, then

$$\frac{\partial\varphi}{\partial y}=0,\ y=-h. \qquad (5)$$

For infinity depth

$$|\nabla\varphi|=0,\ y\to-\infty. \qquad (6)$$

The Initial conditions are

$$\varphi(X,y,0)=\varphi_0(X,y),\ \frac{\partial\varphi}{\partial t}(X,y,0)=\varphi_1(X,y), \quad (7)$$

where φ_0 and φ_1 are known functions.

For unique calculation of wave resistance the body mass and its distribution of body volume or the total mass and the center of mass position must be set.

If the relations (1) and (2) have the nonlinear terms, the numerical solution is only possible. It is obvious that it is useful to have all possible solutions of simpler problems to better understand the problem, to create a theoretical basis and benchmarks for the full nonlinear theory. For this purpose we consider the case when the conditions (1) and (2) are linearized. It is can be valid for weakly submerged hulls or for motion with a relatively high speed, when the immersion is reduced due to the dynamic overpressure.

The boundary conditions (1)—(2) on the solid boundaries after linearization and conditions on the free surface (3)—(4) can be combined, if we introduce a function of the pressure distribution $p(X,t)$ on the entire plane $y=0$, which determines the pressure on the body at $X \in S(t)$ and is zero at the free surface. We will describe the form of fluid boundary by single function $\eta(X,t)$ also, and $\eta(X,t)=f(X,t)$ for $X \in S(t)$.

The new problem can be written as

$$\nabla^2\varphi = 0, \ -h<y<0, \tag{8}$$

$$N\varphi = -gf(X,t) - p(X,t), \ \ y=-0, \tag{9}$$

$$\frac{\partial\varphi}{\partial y} = N\eta(X,t), \ y=-0. \tag{10}$$

We use the Fourier transform to analyze the problems (8)—(10) and (5)—(7).

First, we continue the function φ on the whole space R^{n+1}. We set $\varphi(X,y,t)=0$ for $y>0$ and $\varphi(X,y,t)=\varphi(X,-y-2h,t)$ for $y<-h$.

As a result, the function φ is defined on the whole space and it is continuous with their derivatives at $y=-h$ and it is discontinuous with their derivatives at $y=0$ and $y=-2h$. Then we introduce the generalized functions φ, p and η generated by a classical, over the space of tempered test functions [2]. In this case, the generalized Fourier transforms of these functions exist in the same space.

The generalized function φ obeys the equation

$$\nabla^2\varphi = -\varphi(X,-0,t)[\delta'(y)-\delta'(y+2h)] - \\ -\frac{\partial\varphi}{\partial y}(X,-0,t)[\delta(y)+\delta(y+2h)], \tag{11}$$

where $\delta(y)$ and $\delta'(y)$ are delta function and its derivative. We designate the generalized Fourier transform of function φ in R^n on X as $\Phi(\Lambda,y,t)$, where $\Lambda=(\lambda,\mu)$ in R^2 and $\Lambda=\lambda$ in R^1. We designate also the Fourier transforms of p and η as $P(\Lambda,t)$ and $H(\Lambda,t)$. We obtain

$$\Phi(\Lambda,y,t) = \frac{\cosh|\Lambda|(y+h)}{|\Lambda|\sinh|\Lambda|h}\overline{N}H(\Lambda,t) \tag{12}$$

and relation between $P(\Lambda,t)$ and $H(\Lambda,t)$

$$\left(\frac{1}{\zeta(\Lambda)}\overline{N}^2 + g\right)H(\Lambda,t) = -P(\Lambda,t), \tag{13}$$

by applying the Fourier transform to (11) and the method of fundamental solutions, and taking into account (9) and (10). There are $|\Lambda|=\sqrt{\lambda^2+\mu^2}$ in R^2 and $|\Lambda|=|\lambda|$ in R^1, $\overline{N}=\frac{\partial}{\partial t}+i\lambda V_0$, and $\zeta(\Lambda)=|\Lambda|\tanh|\Lambda|h$. At $h\to\infty$ we have $\zeta(\Lambda)=|\Lambda|$ and $\Phi(\Lambda,y,t)=e^{|\Lambda|y}\overline{N}H(\Lambda,t)/|\Lambda|$.

The general solution of (13) with initial conditions is

$$H(\Lambda,t) = \zeta(\Lambda)\int_0^t P(\Lambda,\tau)e^{i\lambda V_0(\tau-t)}D(\Lambda,\tau-t)d\tau + \\ + G_0(\Lambda,t)e^{-i\lambda V_0 t}, \tag{14}$$

where $D(\Lambda,t)=\left[\sin\sqrt{g\zeta(\Lambda)}t\right]/\sqrt{g\zeta(\Lambda)}$,

$$G_0(\Lambda,t) = H_0(\Lambda)\overline{N}D(\Lambda,t) + H_1(\Lambda)D(\Lambda,t),$$

and the functions $H_0(\Lambda)$ and $H_1(\Lambda)$ are Fourier transforms of functions $\eta_0(\Lambda)=\eta(\Lambda,0)$ and $\eta_1(\Lambda)=\frac{\partial\eta}{\partial t}(\Lambda,0)$ which are obtained from initial conditions. The inverse Fourier transform of (14) in R^n gives the formula for determination of time-dependent shape of the surface generated by the pressure $p(X,t)$ at arbitrary time dependence.

We obtain from (14) the formula for steady motion at a constant speed, assuming that the steady motion appears at larger times when there are no pressure perturbations, that is, when $P(\Lambda,t)\approx P(\Lambda)$. For large t the initial conditions can be set homogeneous and then

$$H(\Lambda,t) = -\frac{i}{2}\sqrt{\frac{\zeta(\Lambda)}{g}}P(\Lambda)\int_{-\infty}^{t}\theta(\tau)\left[e^{-i\tau\xi_1(\Lambda)} - \right. \\ \left. - e^{-i\tau\xi_2(\Lambda)}\right]d\tau,$$

where $\theta(t)$ is the Heaviside function, $\xi_1(\Lambda)=\lambda V_0-\sqrt{g\zeta(\Lambda)}$ and $\xi_2(\Lambda)=\lambda V_0+\sqrt{g\zeta(\Lambda)}$.

At $t\to\infty$, the integral here is the sum of the Fourier transforms of Heaviside functions.

As a result, $H(\Lambda,t)$ will not depend on the time, $H(\Lambda,t)=H(\Lambda)$ and we obtain

$$H(\Lambda) = P(\Lambda)Q(\Lambda) \tag{15}$$

as in [4], where

$$Q(\Lambda) = reg\,\frac{\zeta(\Lambda)}{\lambda^2 V_0^{\,2} - g\zeta(\Lambda)} + \\ + \frac{\pi i}{2}\sqrt{\frac{\zeta(\Lambda)}{g}}\left[\delta(\xi_1(\Lambda)) - \delta(\xi_2(\Lambda))\right],$$

and *reg* indicates a regularization, and delta functions support on a $(n-1)$-dimension surfaces $\xi_1(\Lambda)=0$ and $\xi_2(\Lambda)=0$.

The similar formulas for harmonic oscillations on the wave surface are obtained from (14) based on the assumptions that the pressure becomes harmonic under the action of an regular wave $H_0^*(\Lambda)e^{ikt}$ and $P(\Lambda,t)=P^*(\Lambda)e^{ikt}$, where k is the oscillation frequency, and $H_0^*(\Lambda)$ and $P^*(\Lambda)$ are amplitude functions. At $t\to\infty$, we obtain in analogous, that $H(\Lambda,t)=H^*(\Lambda)e^{ikt}$ and

$$H^*(\Lambda) = P^*(\Lambda)Q^*(\Lambda) + H_0^*(\Lambda). \tag{16}$$

The wave resistance is now

$$R_w(t)=-\int_{R^n} p(X)\cdot\frac{\partial\eta}{\partial x}(X)dX . \qquad (17)$$

We can show that, since the function $p(X)$ is square integrable and finite, and $\frac{\partial\eta}{\partial x}(X)$ is a sectionally continuous, then

$$\int_{R^n} p(X)\frac{\partial\eta}{\partial x}(X)dX=\frac{1}{(2\pi)^n}\int_{R^n}\overline{P(\Lambda)}(i\lambda)H(\Lambda)d\Lambda , \qquad (18)$$

where the overbar means the complex conjugate.
For the motion described by (15), we obtain

$$R_w=-\int_{R^n}|P(\Lambda)|^2(-i\lambda)Q(\Lambda)d\Lambda=$$

$$=-\frac{\pi}{2}\int_{R^n}\lambda\sqrt{\frac{\zeta(\Lambda)}{g}}|P(\Lambda)|^2[\delta(\xi_1(\Lambda))-\delta(\xi_2(\Lambda))]d\Lambda .$$

The products with the delta functions which support by $(n-1)$-dimensional surfaces are calculated by decomposition formulas of generalized functions.
As a result, R_w will depend on the value of $|P(\Lambda)|^2$, calculated at fixed values of Λ, giving the parameters of movement. The value of $|P(\Lambda)|^2$ determines the wave amplitude at infinity and depends on $p(X)$ and S. The minimization of R_w will consist in searching for optimal $p(X)$ and S. We have the analogous formulas in the case of harmonic oscillations in the wave surface.

3. The simple samples

We consider a simple example of 2D theory ($n=1$) of motion with speed V_0 at $h\to\infty$ [3].
Then $\Lambda=\lambda$ and $\xi_{1,2}(\lambda)=\lambda V_0\mp\sqrt{g|\lambda|}$, $\delta(\xi_{1,2}(\lambda))=2\delta(\lambda\mp g/V_0^2)/V_0$, and we have in nondimesional values

$$R_x=\nu|P(\nu)|^2=\nu A^2 , \qquad (18)$$

where $\nu=ga/V_0^2$, a is a linear size, $A=\sqrt{A_s{}^2+A_c{}^2}$, $\tan\alpha=A_s/A_c$,

$$A_s=\int_{-\infty}^{\infty}p(x)\sin\nu x dx , \quad A_c=\int_{-\infty}^{\infty}p(x)\cos\nu x dx .$$

The conditions for mass and center of mass position are

$$\int_{-\infty}^{\infty}p(x)dx=\frac{\Delta_y}{\rho a^2V_0^2}=\nu , \quad \int_{-\infty}^{\infty}xp(x)dx=x_c\nu , \qquad (19)$$

where Δ_y is the body weight, or displacement, and x_c is the center of mass axis, $a=\sqrt[3]{\Delta_y/\rho g}$.
There is the most easily to make a zero amplitude in (18) for symmetrical distribution of pressure at $x_c=0$.

The amplitude of the resulting wave at $p(x)=p_c=const$ on $[-1,1]$ is $A=A_c=p_c\int_{-1}^{1}\cos\nu x dx=2p_c\frac{\sin\nu}{\nu}$, and wave resistance is $R_w=\nu A_c{}^2=\frac{2p_c{}^2}{\nu}\sin^2\nu$. It is equal to zero at $1=\frac{\pi n}{\nu}$, $n=\pm1,\pm2,\dots$. Therefore, the length of the interval $l=2$ must satisfy the condition $l=2\pi n/\nu$, $n=1,2,\dots$. Since the wavelength is $L=2\pi/\nu$, then this condition is $l=nL$. The length of the interval must be a multiple of the wavelength.
We consider the constant pressure distribution on the two symmetrical identical segments $[-1,-b]$ and $[b,1]$ as a

$$p(x)=p_1\vartheta([-1,-b];x)+p_2\vartheta([b,1];x),\ 0<b<1$$

where $\vartheta([c,d];x)$ is the characteristic function of interval which is 1 for $x\in[c,d]$ and is zero for $x\notin[c,d]$. It is ensue from (19) in the case of a symmetric distribution of pressure that must be $p_1=p_2=\frac{\nu}{2(1-b)}$, or

$$p(x)=\frac{\nu}{2(1-b)}\{\vartheta([-1,-b];x)+\vartheta([b,1];x)\} .$$

The calculations show that at $l=2$ the value l/L, giving $R_w=0$, tends to 1 at $b\to0$. It tends to 0.5 at $b\to1$. Indeed, at $b\to1$ we have

$$\lim_{b\to a}p(x)=\frac{\nu}{2}[\delta(x+1)+\delta(x-1)]. \qquad (20)$$

In this case $A_s=0$, and $A_c=\nu\cos\nu$. The wave resistance is zero at $1=(1/2+n)\pi/\nu$, $n=0,\pm1,\pm2,\dots$ or $l=2=(1/2+n)L$, $n=0,1,2,\dots$
The velocity potential for distribution (20) is

$$\varphi(x,y)=$$

$$=\frac{\nu}{2\pi}v.p.\int_0^{\infty}\frac{e^{\lambda y}}{\lambda-\nu}[\sin\lambda(1+x)-\sin\lambda(1-x)]d\lambda+$$

$$+\nu e^{\nu y}\cos\nu\cos\nu x . \qquad (21)$$

We consider also the harmonic oscillations on the wave $\eta_0(x,t)=\mathrm{Re}\,\eta_0{}^*(x)e^{ikt}$. Then

$$H_0{}^*(\lambda)=2\pi[A_0\delta(\lambda-\omega)+B_0\delta(\lambda+\omega)],$$

where A_0 and B_0 are complex constants that specify the wave amplitude, $\omega=ak^2/g$. In this case

$$H^*(\lambda)=reg\frac{P^*(\lambda)|\lambda|}{\omega-|\lambda|}+$$

$$+\pi i\omega[P^*(\omega)\delta(\lambda-\omega)+P^*(-\omega)\delta(\lambda+\omega)]+H_0{}^*(\lambda)$$

and wave resistance is

$$R^*{}_w = i\omega\left[A_0 \overline{P^*(\omega)} - B_0 \overline{P^*(-\omega)}\right] .$$

If the wave is incident on the left then $B_0 = 0$ and

$$R^*{}_w = i\omega A_0 \overline{P^*(\omega)}.$$

Then the modulus of wave resistance

$$\left|R^*{}_w\right| = \omega|A_0|\left|P^*(\omega)\right| = |A_0|^2 R ,$$

where $R = \omega\left|P^*(\omega)\right|/|A_0|$ is the reflection coefficient.

We consider the pressure distribution

$$\operatorname{Re} p^*(x) = \frac{p_c}{2}[\delta(x+1) + \delta(x-1)], \ \operatorname{Im} p^*(x) = 0 .$$

In this case $A_c = p_c \cos\omega$. The reflection coefficient is zero at $\omega = \pi/2 + n\pi$, $n = 0,\pm1,\pm2,...$

The velocity potential in this case is

$$\varphi^*(x,y) = \\ = \frac{\omega p_c}{2\pi} v.p.\int_0^\infty \frac{e^{\omega y}}{\lambda - \omega}[\cos\lambda(1+x) + \cos\lambda(1-x)]d\lambda - \\ - i\omega p_c e^{\omega y}\cos\omega\cos\omega x + \varphi_0{}^*(x,y) \ . \quad (22)$$

Other examples will be shown in the report.

4. Relation with trapped modes

The potentials (21) and (22) for the pairs of symmetric point pressure are there actually limits of potentials of symmetric pairs bodies when they degenerate into points. At zero wave resistance the potentials (21) and (22) do not contain the second terms. The first terms coincide up to a constant factor with potentials that are among the known examples of potentials of trapped modes [4, 5] at $a = 1$.

Trapped modes are defined as modes on the surface of an ideal fluid with a finite energy that do not create waves on infinity, and therefore have zero wave resistance. In the examples these modes are generated by point sources located in a certain way.

Pressure points which we have considered make sense combinations of point sources also and generate waves of finite energy, too. However, the trapped modes [4, 5] have another important characteristic. They are the solutions of the homogeneous boundary value problems for the differences of two potentials and illustrate the examples of non-uniqueness of solutions.

The homogeneous condition on the free surface is also homogeneous in the original problem. In particular, for the problem of Neumann-Kelvin it has the form

$$V_0^2 \frac{\partial^2\varphi}{\partial x^2}(x,-0) + g\frac{\partial\varphi}{\partial y}(x,-0) = 0 . \qquad (23)$$

This condition is also satisfied for the difference of potentials. The homogeneous condition on the solid body is $\partial\varphi/\partial n = 0$. However, in the case of degeneracy of the rigid body to the point this condition makes no sense in the classical meaning.

The contours of the streamlines covering the singularities, considered as the contours of bodies for constructing examples of trapped modes. The condition $\partial\varphi/\partial n = 0$ is satisfied on these contours due to the fact that they are free streamlines inside fluid and on which have no pressure drop.

In general, it is possible to build such kind of trapped modes for a given contour or configuration of bodies by a selection of parameters and geometry of singularities. However any condition for $\partial\varphi/\partial n$ does not make sense for the classical functions at points of non-smooth contact of body and fluid boundaries. We may notice that the condition (23) is satisfied on the boundary with zero pressure, and on the boundary with a nonzero constant pressure excepting the boundary points. This condition is also indefinite at the points of singularities location when they are distributed on the free surface. In all cases, the boundary conditions in the classical formulation of problems are satisfied everywhere except for a set of points of function discontinuities and their derivatives. The non-uniqueness of solutions of boundary value problems in the classical formulation may be a result of indefinite characteristics of singularities on these sets of breaks and their geometry.

The use of generalized functions eliminates the problem of non differentiability of classical functions. The correct interpretation in generalized functions of the physical meaning of discontinuities of functions is needed for studying and resolving the problem of non-uniqueness of solutions of boundary value problems in the linear wave theory.

References

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