The effect of a vertically sheared current on rogue wave properties.

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Rogue waves are unexpected large and spontaneous waves that can occur on the surface of calm sea and during severe storm. In this study a nonlinear Schrodinger equation governing the complex envelope of a surface water wave train propagating on an uniform vertical shear current is used (Thomas R., Kharif C. & Manna M., 2012). As a result it is shown that the vorticity modifies significantly the properties of rogue wave, namely their lifetime and height. In the presence of a shear current co-flowing with the waves it is shown that the height of the rogue waves is increased whereas the lifetime is decreased. Just the opposite, for a counter-flowing current, the height and lifetime are reduced and increased, respectively.

I. INTRODUCTION

Rogue waves are among the wave naturally observed by people on the sea surface that represent an inseparable feature of the Ocean. Rogue waves appear from nowhere, cause danger, and disappear at once. They may occur on the surface of a relatively calm sea and not reach very high amplitudes, but still be fatal for ships and crew due to their unexpectedness and abnormal features. A wave is considered as a rogue wave if its height H_r is more than twice the significant height H_s. For a Gaussian sea and a narrow band spectrum, the significant height $H_s = 4\sigma$, where σ is the standard deviation of the elevation. Rogue wave can be generated by different mechanisms such as wave-current interaction, geometrical or dispersive focusing, modulation instability (the Benjamin-Feir instability), soliton collison, crossing seas, etc. In this study we consider rogue waves due to modulational instability. Several studies have been carried out on the propagation of surface water waves propagating steadily on a rotational current ([2, 5, 6]). Few papers have been published on the effect of a vertical shear current on the Benjamin-Feir instability of a Stokes' wave train in the presence of uniform vorticity. Johnson [3] studied the slow modulation of a harmonic wave on a two dimensional flow of arbitrary vorticity. Using the method of multiple scale he obtained the condition of linear stability for a plane nonlinear wave. This condition is verified if the product of the dispersive and nonlinear terms of the nonlinear Schrodinger equation (NLS equation) is negatif. The instability properties of weakly nonlinear wave packet to three dimensional perturbations have been studied by Oikawa *et al.* [4]. Their analysis were illustrated for the case of a linear shear. The Benjamin-Feir instability of a wave train propagating on positive and negative shear currents have been studied by Choi [1]. For a fixed steepness, he found that the envelope of the modulated wave train grows faster in a positive shear current and slower in a negative one. Thomas *et al.* [7] using the method of multiple scales derived a vor-NLS equation in finite depth when the vorticity is taken into account. They carried out a stability analysis of a weakly nonlinear wave train in the presence of uniform vorticity. They demonstrated that vorticity modifies significantly the modulational instability properties of weakly nonlinear plane waves, namely the growth rate and nabwidth. They shown that these plane wave solutions may be linearly stable to modulational instability independently of the dimensionless parameter kh. Using the Benjamin-Feir Index (BFI) concept, they demonstrated that the number of rogue waves increases in the presence of a shear current co-flowing with the wave whereas it is the opposite for a counter-flowing current. We consider the vor-NLS equation derived by Thomas *et al.* [7] to investigate the properties of rogue waves in the presence of vertical uniform shear current, namely their lifetime and amplification.

II. THE VOR-NLS EQUATION

Generally in coastal and ocean waters, the velocity profiles are varying with depth due to the bottom friction and surface wind stress. For example, tide currents may have an important effect on the propagation of waves and wave pack-

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ets. Surface drift of the water induce by wind can also affect their propagation due to the velocity in the surface layer. The velocity field can be decomposed into a rotational term and an irrotational term which correspond to the induced wave velocity (the wave motion is assume to be potential). The fluid is inviscid and incompressible. Hence, the Kelvin-Lagrange theorem states that the vorticity is conserved.

$$\mathbf{V} = \Omega y \mathbf{i} + \nabla \phi(x, y, t) \tag{1}$$

The wave train move along the x axis, the y axis is oriented upward and gravity downward. The water depth h is constant and the bottom is located at $y = -h$. Ω represents the magnitude of the shear and ϕ the velocity potential due to the wave.

A. Governing equations

The governing equtions are :
$\nabla^2 \phi = 0, -h < y < \eta(x, t)$
$\phi_y = 0, y = -h$
$\eta_t + (\Phi_x + \Omega\eta)\eta_x - \Phi_y = 0$
$\Phi_t + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_y^2 + \Omega\eta\Phi_x + g\eta - \Omega\Psi = 0$
where Φ means that ϕ is calculated on the free surface and η the free surface elevation. Using the Cauchy-Riemann conditions, the dynamic boundary condition may be rewritten as follows :

$$\Phi_{tx} + \Phi_{ty}\eta_x + \Phi_x(\Phi_{xx} + \Phi_{xy}\eta_x) + \Phi_y(\Phi_{xy} + \Phi_{yy}\eta_x)$$
$$+\Omega\eta_x\Phi_x + \Omega\eta(\Phi_{xx} + \Phi_{xy}\eta_x) + g\eta_x + \Omega(\Phi_y - \Phi_x\eta x = 0$$

B. The multiple scale analysis

We present briefly the derivation of the NLS equation by using the method of multiple scales (for more details see Thomas & al. 2012 [7]).

$$\phi = \sum_{n=-\infty}^{+\infty} \phi_n exp[in(kx - \omega t)], \tag{2}$$

$$\eta = \sum_{n=-\infty}^{+\infty} \eta_n exp[in(kx - \omega t)], \tag{3}$$

where k is the wavenumber of the carrier and ω its frequency. Then ϕ_n and η_n are written in perturbation series

$$\phi_n = \sum_{j=n}^{+\infty} \epsilon^j \phi_{nj}, \tag{4}$$

$$\eta_n = \sum_{j=n}^{+\infty} \epsilon^j \eta_{nj}, \tag{5}$$

with ϵ the small parameter corresponding to the wave steepness of the carrier.

We seek a solution modulated on a slow time scale $\tau = \epsilon^2 t$ and slow space scale $\xi = \epsilon(x - c_g t)$

$$\eta(x, t) = \frac{1}{2}(\epsilon a(\xi, \tau)\exp[i(kx - \omega t)] + c.c) + \mathcal{O}(\epsilon^2) \tag{6}$$

With c_g the group velocity of the carrier wave. The new system of governing equations becomes :

$$\begin{aligned}
\epsilon^2\phi_{\xi\xi} + \phi_{yy} = 0, -h < y < \eta(x, t)\\
\phi_y = 0, y = -h\\
\epsilon^2\eta_\tau - \epsilon c_g \eta_\xi + \epsilon^2\Phi_\xi\eta_\xi + \epsilon\Omega\eta\eta_\xi - \Phi_y = 0\\
\epsilon^3\Phi_{\xi\tau} - \epsilon^2 c_g\Phi_{\xi\xi} + \epsilon^3\Phi_{y\tau}\eta_\xi - \epsilon^2 c_g\Phi_{\xi y}\eta_\xi\\
+ \epsilon^3\Phi_\xi\Phi_{\xi\xi} + \epsilon^3\Phi_\xi\Phi_{\xi y}\eta_\xi + \epsilon\Phi_y\Phi_{\xi y} + \epsilon\Phi_y\Phi_{yy}\eta_\xi\\
+ \epsilon^2\Omega\eta_\xi\Phi_\xi + \epsilon^2\Omega\eta\Phi_{\xi\xi} + \epsilon^2\Omega\eta\Phi_{\xi y}\eta_\xi + \epsilon g\eta_\xi\\
+ \Omega\Phi_y - \epsilon^2\Omega\Phi_\xi\eta_\xi = 0
\end{aligned}$$

Substituing the expansion for the potential ϕ and the free surface elevation η lead to the nonlinear Schrodinger equation (after a tedious development to the third order) :

$$ia_\tau + La_{\xi\xi} + N|a|^2 a = 0 \tag{7}$$

where

$$\begin{aligned}
L &= \frac{\omega}{k^2\sigma(2 + X)}\mu(1 - \sigma^2)[1 - \mu\sigma + (1 - \rho)X] - \sigma\rho^2\\
M &= \frac{-\omega k^2(U + VW)}{8(1 + X)(2 + X)\sigma^4}\\
U &= 9 - 12\sigma^2 + 13\sigma^4 - 2\sigma^6 + (27 - 18\sigma^2\\
&\quad + 15\sigma^4)X + (33 - 3\sigma^2 + 4\sigma^4)X^2 + (21 + 5\sigma^2)X^3\\
&\quad + (7 + 2\sigma^2)X^4 + X^5\\
V &= (1 + X)^2(1 + \rho + \mu\overline{\Omega}) + 1 + X - \rho\sigma^2 - \mu\sigma X\\
W &= 2\sigma^3\frac{(1 + X)(2 + X) + \rho(1 - \sigma^2)}{\sigma\rho(\rho + \mu\overline{\Omega}) - \mu(1 + X)}
\end{aligned}$$

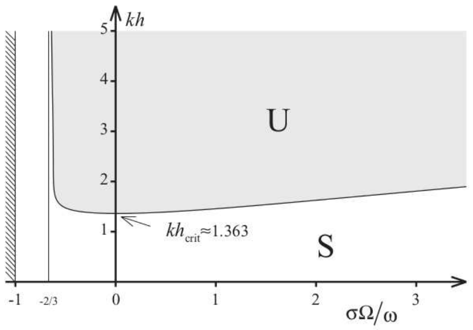


FIG. 1: *Stability diagram.* ***S****: stable,* ***U****: unstable. (Thomas et al. (2012) [7])*

with

$$\begin{aligned} \mu &= kh \\ \sigma &= \tanh(kh) \\ \rho &= \frac{c_g}{c_p} \\ \overline{\Omega} &= \frac{\Omega}{\omega} \\ X &= \sigma\overline{\Omega} \end{aligned}$$

III. STABILITY ANALYSIS

A solution of equation (7) can be expressed using a Stoke wave :

$$a = a_0 exp(iNa_0\tau) \tag{8}$$

A linear stability analysis gives the stability criterion for a Stoke's wave :

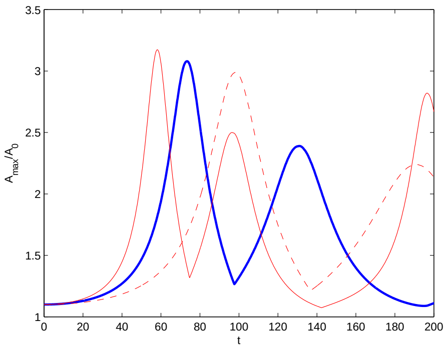
$$L(-2Na_0^2 + l^2L) \geq 0 \quad \text{stable} \tag{9}$$
$$L(-2Na_0^2 + l^2L) < 0 \quad \text{unstable} \tag{10}$$

The stability domain is plotted in figure 1 as a function of $\sigma\overline{\Omega}$ and kh.

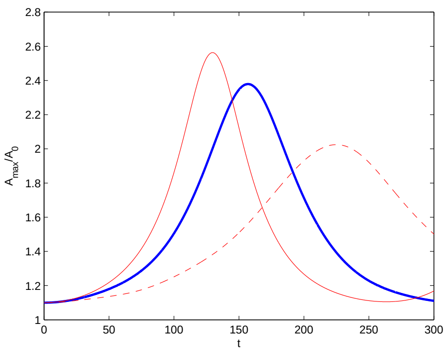
There are two critical values as a function of the vorticity and the water depth. Indeed for a value of $\sigma\overline{\Omega} < -2/3$ which correspond to a vorticity $\Omega = -2\sqrt{\frac{kg}{3}}$, Stoke's waves are stable. When there is no vorticity ($\overline{\Omega} = 0$) Stoke's waves are stable for $kh < 1.363$. Note that for three dimensional motion there are oblique modulations even when $kh < 1.363$.

IV. APPLICATION TO ROGUE WAVES

In the previous section a development to third order and a stability analysis have been performed to investigate the influence of the vorticity on the Benjamin-Feir instability of Stoke's waves (see [7] for more details). In this section we consider the nonlinear evolution of the unstable infinitesimal perturbations within the framework of the vor-NLS equation. A series of numerical simulations of the vor-NLS equation is performed for different values of the wave steepness of the carrier wave, the water depth and the vorticity. The nonlinear stability analysis is developed for constant vorticity varying from -0.4 to 0.4 and for $kh = \infty$ or $kh = 2$. Due to the limitation of the NLS equation to weakly nonlinear wave trains we choose $ak = 0.05$ and $ak = 0.10$. The wave number of the carrier wave is $k = 10$ and that of the perturbation is $l = 1$.



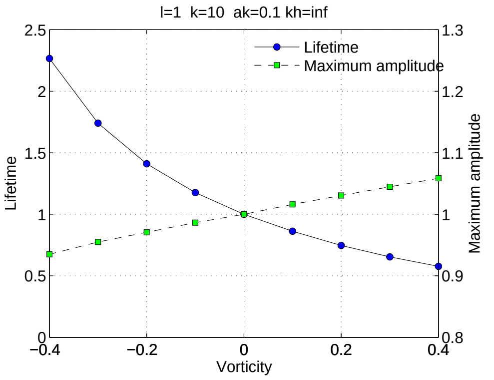
(a)



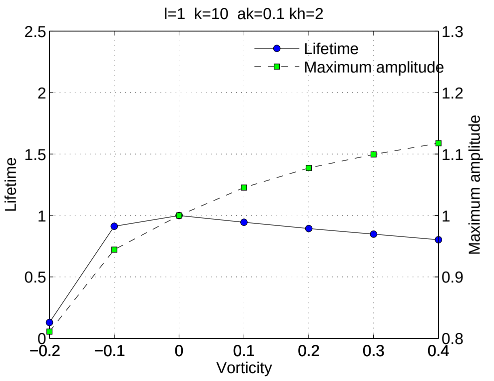
(b)

FIG. 2: *Evolution of the normalized maximum amplitude as a function of time. In infinite (a) and finite (b) depth. Thick continuous line corresponds to* 0 *vorticity, thin continuous one to* 0.2 *and thin dashed line to* -0.2.

The normalized maximum amplitude evolution as a function of time is shown in figure 2 for three different values of the vorticity (-0.2, 0 and 0.2). In both, finite and infinite depth, increasing the vorticity increases the maximum amplitude of the perturbation but decreases the width of the peak. To have a rogue wave event, the lifetime has been evaluated for a amplitude higher than two times the initial amplitude of the Stoke's wave.



(a)



(b)

FIG. 3: *Evolution of the life time and the maximum amplitude as a function a vorticity in infinite (a) and finite (b) depth.*

Figure 3 shows the evolution of the lifetime and normalized maximum amplitude of the envelope as a function of the vorticity. In infinite (fig.3.a) and finite depth (fig.3.b) increasing the vorticity, increases the maximum amplitude but decreases the lifetime of the rogue wave. In figure 3.b the Stoke's wave is stable for values of vorticity lower than -0.2. This explain why the lifetime and maximum amplitude for negative values of vorticity are lower than those without vorticity, the instability is just high enough to be defined as a rogue wave.

V. CONCLUSION

Using a 1D nonlinear Schrodinger equation in the presence of a vertical shear current of non zero constant vorticity in finite and infinite depth, we have shown that the lifetime and maximum amplitude of rogue waves are significantly influenced by a vertical shear current. For a counter-flowing current the lifetime of a rogue is increased whereas the height is decreased. The opposite situation is observed for a co-flowing current. This study is still in progress, to take into account the influence of wind by introducing the Miles' mechanism.

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