

# The Wave Radiation Problem in A Two-layer Fluid by Time-domain Method

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## INTRODUCTION

In the real ocean, density of sea water is actually changing with the depth due to the variations in salinity and temperature. The two-layer fluid system is the simplest model to investigate the internal wave. In this system there exists a density discontinuity at the interface between the upper and the lower layers, and the density is constant in each layer. The internal waves will be generated on the interface such as solitary wave and periodic wave. For internal solitary wave, the wave length is much longer than the characteristic length of structure. So the internal solitary wave forces acting on ocean structures can be simulated by Morison formula (Song, et al. 2011). For the harmonic internal waves, the wave length is over a wide range. The diffraction/radiation theory should be adopted when the characteristic length of structure is relative large.

The multipole expansion method is utilized by Cadby and Linton (2000) to investigate wave radiation on a submerged sphere. Ten and Kashiwagi (2004) and Kashiwagi et al. (2006) developed a linearized 2-D diffraction/radiation model, in which boundary integral-equation method is implemented in frequency-domain. In this model, the Green function which satisfied the free surface and interface boundary conditions is derived. Nguyen and Yeung (2011) derived the unsteady 3-D sources for a two-layer fluid of finite depth.

Teng and Gou(2009) and Gou et al. (2012) developed a time-domain numerical method by boundary element method to study the diffraction problem in a two-layer fluid. In this numerical model, the simple Rankine source is used, and the complicated wave Green function is avoided to be calculated. As the continued work, the wave radiation problem in a two-layer fluid is simulated by this time-domain method here. The forced oscillation of a truncated cylinder in finite depth is considered. Comparisons are made with an analytic solution derived by Shi and You (2009), which the eigenfunction expansion method is used. The examination shows that the model gives very steady

results and has good agreement with the analytic solution, and some significant results can be drawn.

## NUMERICAL MODEL

A Cartesian coordinate system is defined with the origin in the plane of the undisturbed free surface, and the z-axis positive upwards. The densities of the fluids in the upper and the lower layers are  $\rho_1$  and  $\rho_2$ , respectively. Other notations are shown in Fig.1. The fluid in each layer is assumed to be inviscid and incompressible, and the flow irrotational. Furthermore, the motions of the body are relatively small so that the linear potential theory is applied.

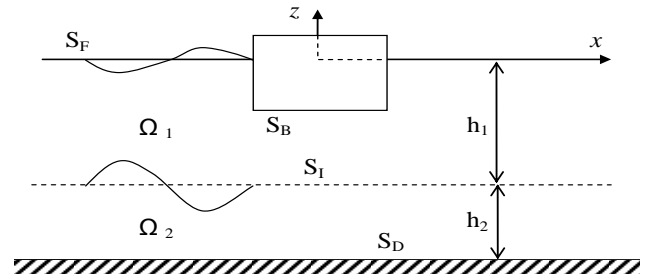


Fig.1 Definition Sketch

The radiation velocity potentials  $\Phi^{(1)}$  and  $\Phi^{(2)}$  in the fluid domain  $\Omega_1$  and  $\Omega_2$  satisfy Laplace equation, and the linearized boundary conditions are satisfied as follows:

$$\Phi^{(1)}_{,tt} + g\Phi^{(1)}_{,z} = 0, \quad \text{on } z=0 \quad (1)$$

$$\Phi^{(1)}_{,z} = \Phi^{(2)}_{,z}, \quad \text{on } z=-h_1 \quad (2)$$

$$\gamma(\Phi^{(1)}_{,tt} + g\Phi^{(1)}_{,z}) = \Phi^{(2)}_{,tt} + g\Phi^{(2)}_{,z}, \quad \text{on } z=-h_1 \quad (3)$$

$$\frac{\partial \Phi^{(1)}}{\partial n} = V_n \quad (4)$$

$$\Phi^{(2)}_{,z} = 0, \quad \text{on } z=-h_1-h_2 \quad (5)$$

where  $\gamma = \rho_1 / \rho_2$ ,  $V_n$  is the velocity of the body.

In order to avoid the reflection of scatter waves, an

artificial damping layer must be utilized to absorb the scattered wave. For the two-layer fluid system, a damping term is added to both the free surface and the internal surface boundary conditions on the outer part.

$$\frac{\partial \eta^{(m)}}{\partial t} = \frac{\partial \Phi^{(m)}}{\partial z} - \nu(r)\eta^{(m)} \quad m = 1, 2 \quad (6)$$

$$\frac{\partial \Phi^{(m)}}{\partial t} = -g\eta^{(m)} - \nu(r)\Phi^{(m)} \quad m = 1 \quad (7)$$

$$\frac{\partial \varphi}{\partial t} = (1 - \gamma)g\eta^{(m)} - \nu(r)\varphi \quad m = 2$$

where

$$\varphi = \gamma\Phi^{(1)} - \Phi^{(2)} \quad (8)$$

$$\nu(r) = \begin{cases} \alpha\omega \left( \frac{r - r_0}{\lambda} \right)^2 & (r_0 \leq r \leq r_0 + \beta\lambda) \\ 0 & (r < r_0) \end{cases} \quad (9)$$

Here, damping coefficient  $\alpha$  and beach breadth coefficient  $\beta$  are equal to 1.0.

The structure considered here is located in the upper fluid. Applying the Green's second identity to Green function and radiation velocity potential  $\Phi^{(1)}$  and  $\Phi^{(2)}$  in each layer respectively, we can obtain the integral equations:

$$\begin{aligned} \alpha\Phi^{(1)} - \iint_{S_B} \Phi^{(1)} \frac{\partial G_1}{\partial n} ds + \iint_{S_F} G_1 \frac{\partial \Phi^{(1)}}{\partial n} ds + \iint_{S_I} G_1 \frac{\partial \Phi^{(1)}}{\partial n} ds \\ = -\iint_{S_B} G_1 \frac{\partial \Phi^{(1)}}{\partial n} ds + \iint_{S_F} \Phi^{(1)} \frac{\partial G_1}{\partial n} ds + \iint_{S_I} \Phi^{(1)} \frac{\partial G_1}{\partial n} ds \end{aligned} \quad (10)$$

$$\iint_{S_I} G_2 \frac{\partial \Phi^{(2)}}{\partial n} ds = \iint_{S_I} \Phi^{(2)} \frac{\partial G_2}{\partial n} ds - \alpha\Phi^{(2)} \quad (11)$$

where

$$G_1 = -\frac{1}{4\pi r}, \quad G_2 = -\frac{1}{4\pi r} - \frac{1}{4\pi r_2} \quad (12)$$

Here,  $r$  is the distance between the field and the source points, and  $r_2$  is the distance between the field point and the image of the source point about the sea bed. The direct method is used to calculate the solid angle  $\alpha$  (Teng, et al, 2006). The computational field is divided by second order isoparametric elements, and two sets of linear equations are obtained after the discretization to Eqs.(10) and (11).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_F} \\ \left\{ \Phi^{(1)} \right\}_{S_B} \\ \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_I} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \Phi^{(1)} \right\}_{S_F} \\ \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_B} \\ \left\{ \Phi^{(1)} \right\}_{S_I} \end{Bmatrix} \quad (13)$$

$$[b] \begin{Bmatrix} \frac{\partial \Phi^{(2)}}{\partial n} \end{Bmatrix}_{S_I} = [t] \left\{ \Phi^{(2)} \right\}_{S_I} \quad (14)$$

Applying the interface conditions Eq.(2) and the velocity potential function  $\varphi$  defined by Eq.(8), we can get a single set of linear equations as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} + \frac{1}{\gamma} s_{13} t^{-1} b \\ a_{21} & a_{22} & a_{23} + \frac{1}{\gamma} s_{23} t^{-1} b \\ a_{31} & a_{32} & a_{33} + \frac{1}{\gamma} s_{33} t^{-1} b \end{bmatrix} \begin{Bmatrix} \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_F} \\ \left\{ \Phi^{(1)} \right\}_{S_B} \\ \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_I} \end{Bmatrix} = \quad (15)$$

$$\begin{bmatrix} s_{11} & s_{12} & \frac{1}{\gamma} s_{13} \\ s_{21} & s_{22} & \frac{1}{\gamma} s_{23} \\ s_{31} & s_{32} & \frac{1}{\gamma} s_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \Phi^{(1)} \right\}_{S_F} \\ \left\{ \frac{\partial \Phi^{(1)}}{\partial n} \right\}_{S_B} \\ \left\{ \varphi \right\}_{S_I} \end{Bmatrix}$$

From the above equation, we can find that the velocity potential  $\Phi^{(1)}$  and  $\Phi^{(2)}$  on the interface have no need to be solved in the time matching procedure, and they have been replaced by  $\varphi$ . It makes the time matching process easier. The 4<sup>th</sup>-order Runge-Kutta approach is used, basing on boundary conditions Eqs.(4), (6) and (7). In order to avoid the initial effect a ramping function is utilized.

$$R_m(t) = \begin{cases} \frac{1}{2} (1 - \cos(\frac{\pi t}{T_m})) & t \leq T_m \\ 1 & t \geq T_m \end{cases} \quad (16)$$

where  $T$  is the wave period, and  $T_m=2T$  in this simulation. Then, all the time histories of wave elevations on free and internal surface and wave forces acting on the body can be calculated.

## NUMERICAL RESULTS

The forced oscillation of a truncated cylinder in a two layer fluid of finite depth is considered. The sketch is also shown in Fig.1. The cylinder has a radius of  $a=1.0\text{m}$ , and a draft of  $T=1.0\text{m}$ . The water depths of the upper and the lower layers are  $h_1=1.4\text{m}$ ,  $h_2=0.6$ . The densities of the fluids in the upper and the lower layers are  $\rho_1=998.2\text{kg/m}^3$  and  $\rho_2=1027.2\text{kg/m}^3$ , respectively. That means  $\gamma=0.972$ .

In this numerical method, both the free surface and internal surface should be meshed. The radius of computed field is about  $3\lambda$  in this case, and the length of the damping layer  $1.5\lambda$  is included. Here,  $\lambda$  is the wave length relative to the internal wave mode. The computational meshes on the body surface, the free surface and the internal surface are shown in Fig.2

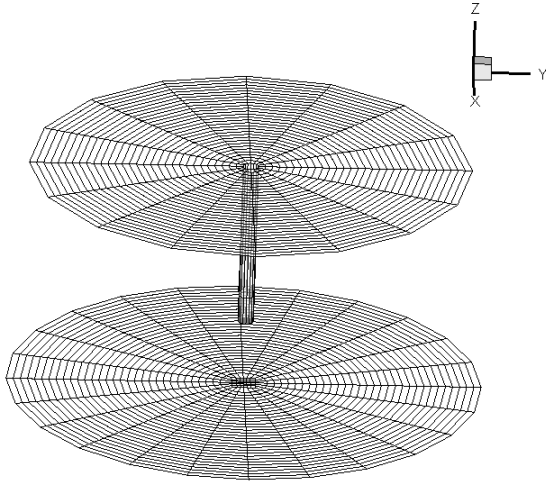


Fig.2 Mesh of the computational field.

### Surge motion

The truncated cylinder is forced to oscillate in surge direction by the function of  $\xi_x=A\sin\omega t$ .  $A$  is the amplitude of the surge oscillation, and  $\omega$  is the oscillation angle frequency. So the velocity of the truncated cylinder  $V_x=A\omega \cos\omega t$ . When the unknown velocity potentials over truncated cylinder surface are obtained, the pressure can be derived from the Bernoulli equation, the forces and moments acting on the body can be calculated by integrating the pressure over the mean body surface. In order to compare with the analytical result derived by Shi and You (2009), the radiation force of unit oscillation was divided into added mass and damping coefficient. In order to investigate the hydrodynamic forces in two-layer fluid system, the results of single-layer is also shown.

Fig.3 and Fig.4 show the added mass and damping coefficient of surge oscillation. It should be note that there are two possible scatter waves with a prescribed frequency in a two-layer fluid, which so called the surface wave mode and internal wave mode. The

results presented here are the total action of both the two modes. From the comparison with analytic results, we can see that the present numerical model by time-domain method have good precision. Furthermore, the results indicate that the results of two-layer fluid are significant compared with the results of single-layer in relative lower frequency range. The interface elevation at  $t=9T$  is shown in Fig.5. It can be seen that the wave profile is smooth enough, and demonstrated the steady of this numerical model.

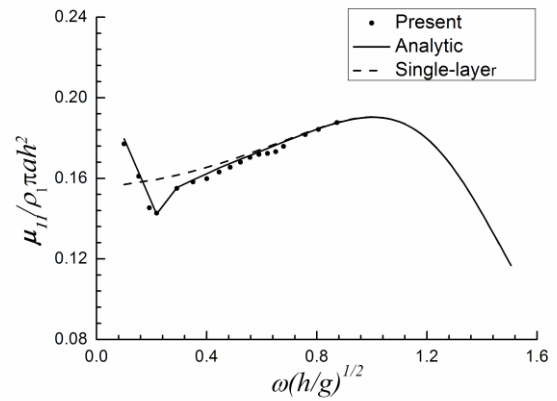


Fig.3 Added mass of surge motion

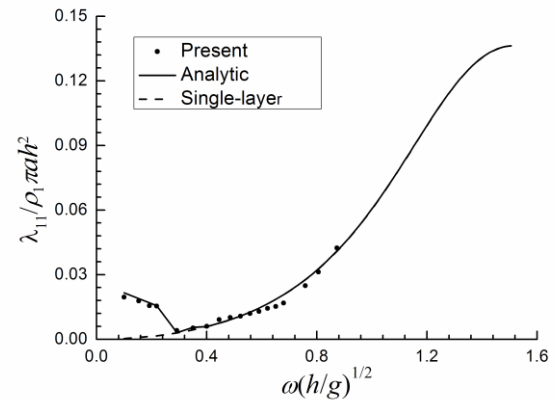


Fig.4 Damping coefficient of surge motion

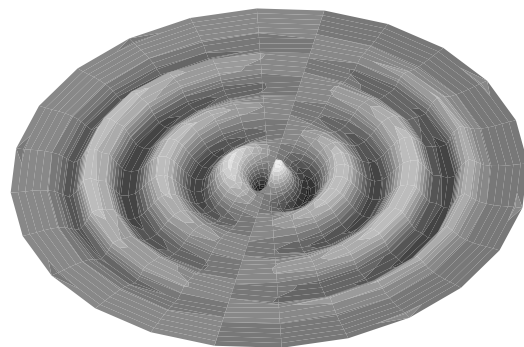


Fig.5 The interface elevation at  $t=9T$

## Pitch motion

The truncated cylinder is also forced to oscillate in pitch direction by the function of  $\alpha_y = A \sin \omega t$ . Here,  $A$  and  $\omega$  are the amplitude and angle frequency of the pitch oscillation, respectively. The rotation center is located on the origin center.

Fig.6 and Fig.7 show the added mass and damping coefficient of pitch motion, respectively. Consistent with previous conclusions, the numerical results have good agreement with analytic solutions, and the results are significant in the relative lower frequency range compared with the single-layer solutions.

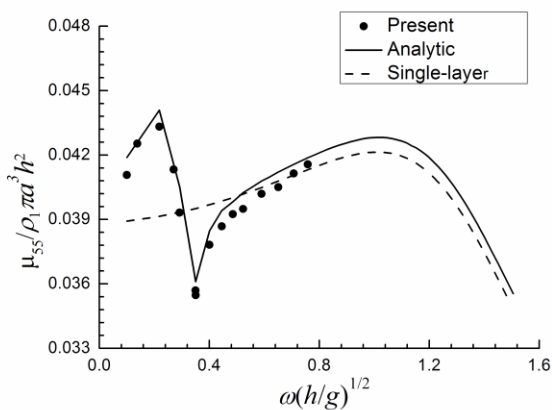


Fig.6 Added mass of pitch motion

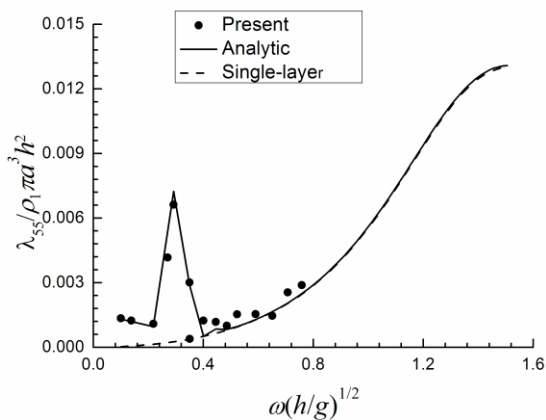


Fig.7 Damping coefficient of pitch motion

## CONCLUSIONS

A boundary element method was developed to solve the linearized wave radiation problem in a two-layer fluid system. The simple Rankine source is used to form the boundary integral equation in time-domain, so the present method can be applied to study the instant wave diffraction and radiation problem. However, more computational cost is needed in this method, because of both free and internal surface must be meshed.

As an example of the computed results, the forced oscillations of a truncated cylinder in surge and pitch directions were considered, respectively. From the comparison with the analytical solutions, it was confirmed that the present numerical model can be applied to investigate the wave radiation problem in a two-layer fluid. By the comparison with single-layer fluid, the results show the significant in relative lower frequency range.

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