

# The Limits of Applicability of Shallow-Water Wave Theory

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## Summary

The shallow-water wave theory of Tuck (1966) provides a remarkably simple set of formulas for the sinkage, trim, and wave resistance of a slender vessel traveling at a steady speed. Tuck (1967) later extended this work to include the case of a channel of finite width. In the current study, we compare these predictions of Tuck with the exact finite-channel-width finite-water-depth linearized theory based on the disturbance of the free surface, as detailed by Doctors (2008). It is shown that the shallow-water theory provides a good approximation for all three abovementioned quantities at speeds up to those corresponding to a depth Froude number of about 0.6. However, the shallow-water theory always provides underpredictions, compared to both the exact theory and with experiments in a towing tank.

## 1 Introduction

We consider the problem of a ship with length  $L$ , beam  $B$ , and draft  $T$ , advancing with a steady speed  $U$  in water of finite depth  $d$  in a channel of width  $w$ . Principal matters of interest are the sinkage  $s$ , trim  $t$ , and resistance  $R$  of the vessel. Of particular concern is whether the vessel will “ground” or contact the bottom in very shallow channels. Thus, it is useful to compute the keel clearance  $c$ , when the vessel is underway. Details of this geometry are shown in Figure 1(a).

Michell (1898) was the first to analyze the hydrodynamics of a ship, although his work was limited to the case of water of infinite depth and with infinite lateral extent. His essential assumption regarding the physics is that the vessel should be “thin”; that is,  $B/L$  is to be small. Many comparisons of the theory with experiments in a towing tank confirm that the predictions possess engineering validity for vessels with  $B/L < 0.25$ .

Regarding wave resistance, this theory has since been extended to include the case of a channel with a finite width, by Sretensky (1936). Finite water depth was considered by Lunde (1951).

If one is concerned with sinkage and trim, then it is necessary to also compute the rather more complicated *near-field* integrals, as described by Doctors (2008). Thus, it is of interest to compare the efficacy of the simpler shallow-water theory with that of the complete finite-depth theory.

## 2 Finite-Water-Depth Wave Theory

A summary of the inviscid linearized theory predicting near-field disturbance created by a steadily advancing ship was provided by Doctors (2008). The wave elevation is given by:

$$\begin{aligned} \zeta(x, y) = & \frac{1}{\pi^2} \int_0^\infty dk_x \sum_{i=0}^\infty \epsilon \Delta k_y k_x^2 \exp(ik_x x) \\ & \cdot \cos(k_y y) \cdot (\mathcal{U} - i\mathcal{V})/f \\ & - \frac{i}{\pi} \sum_{i=0}^\infty \epsilon \Delta k_y k k_x \exp(ik_x x) \\ & \cdot \cos(k_y y) \cdot (\mathcal{U} - i\mathcal{V})/\frac{df}{dk}, \end{aligned} \quad (1)$$

where the dispersion relationship and its derivative are

$$f = k^2 - k k_0 \tanh(kd) - k_y^2, \quad (2)$$

$$\frac{df}{dk} = 2k - k_0 \tanh(kd) - k k_0 d \operatorname{sech}^2(kd), \quad (3)$$

and the fundamental circular wave number is

$$k_0 = g/U^2. \quad (4)$$

Here,  $g$  is the acceleration due to gravity,  $k_x$  is the longitudinal wavenumber,  $k_y$  is the transverse wavenumber and  $k$  is the circular wavenumber. Finally, the finite-depth wave functions in Equation (1) are given by the formulas

$$\mathcal{U} = \frac{P^+ + \exp(-2kd)P^-}{1 + \exp(-2kd)}, \quad (5)$$

$$\mathcal{V} = \frac{Q^+ + \exp(-2kd)Q^-}{1 + \exp(-2kd)}, \quad (6)$$

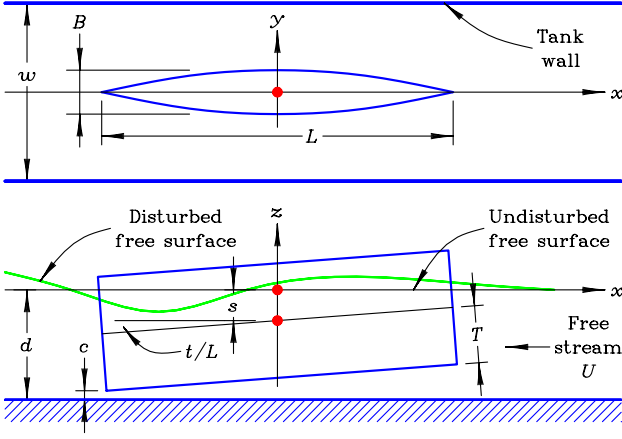


Figure 1: Definition of Problem  
(a) Details of Geometry

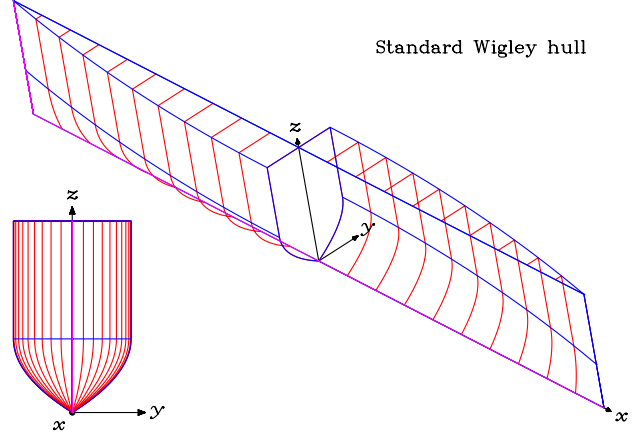


Figure 1: Definition of Problem  
(b) Wireframe View

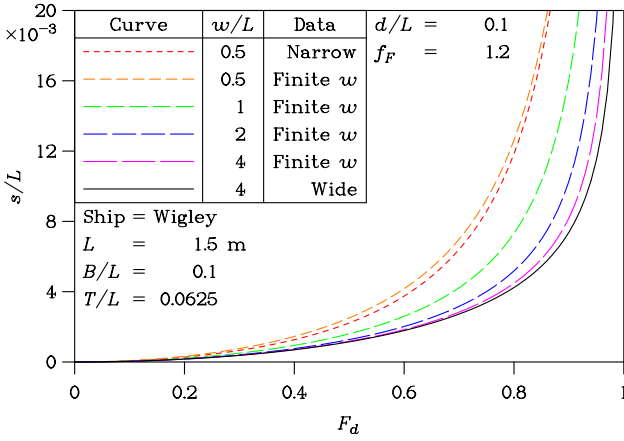


Figure 2: Three Theories for Sinkage

in which the Michell (1898) deep-water monohull functions depend on the local beam  $b(x, z)$ :

$$P^\pm + iQ^\pm = \int_{S_0} b(x, z) \exp(ik_x x \pm kz) dS. \quad (7)$$

### 3 Shallow-Water Wave Theory

The approach here is to assume that the water-depth is sufficiently small, so that the Laplace (field) equation,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad (8)$$

can be replaced by the simpler

$$(1 - F_d^2)\phi_{xx} + \phi_{yy} = 0, \quad (9)$$

where the depth Froude number is

$$F_d = U/\sqrt{gd}. \quad (10)$$

Tuck (1967) showed, by applying the  $x$ -wise Fourier transforms to Equation (9), that the wave elevation could be expressed as

$$\zeta(x, 0) = -\frac{U^2}{4\pi g d \sqrt{1 - F_d^2}} \int_{-\infty}^{\infty} \exp(ikx) |k| \cdot \tilde{S}(k) \coth\left(\frac{1}{2}w|k|\sqrt{1 - F_d^2}\right) dk, \quad (11)$$

where the Fourier transform of the sectional-area curve is

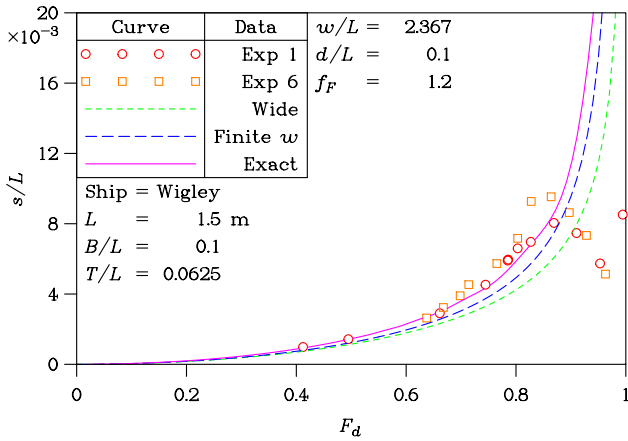
$$\tilde{S}(k) = \int_L S(x) \exp(-ikx) dx. \quad (12)$$

The pressure at the local section is directly related to the wave elevation, so that one can compute the relevant contributions to the longitudinal  $x$  force, the vertical  $z$  force, and the moment about the transverse  $y$  axis. The resistance, sinkage, and trim directly follow.

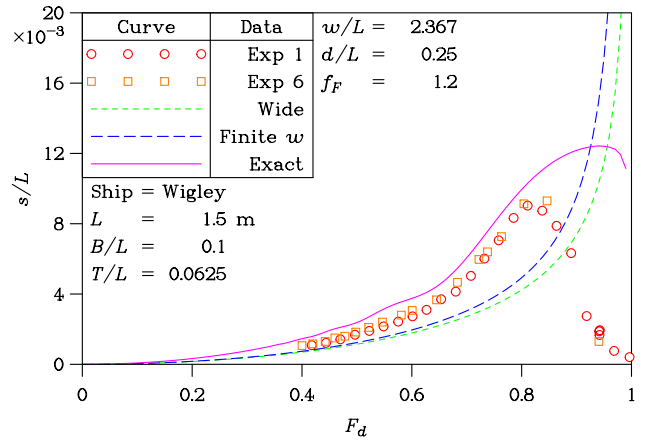
### 4 Sinkage

The subject of the experiments was a “simple” model possessing parabolic sections and parabolic waterlines, as suggested by Wigley (1934). The model is shown in Figure 1(b).

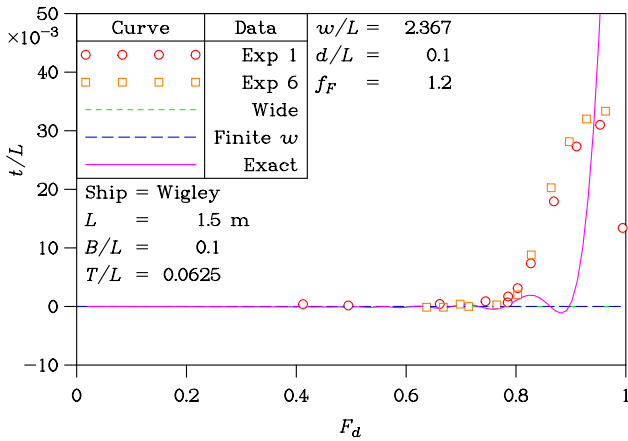
Figure 2 is a comparison of the theoretical dimensionless sinkage  $s/L$  for this model, plotted as a function of the depth Froude number  $F_d$ . Curve 2 through Curve 5 represent the finite-width predictions, based on Equation (11). It is seen that



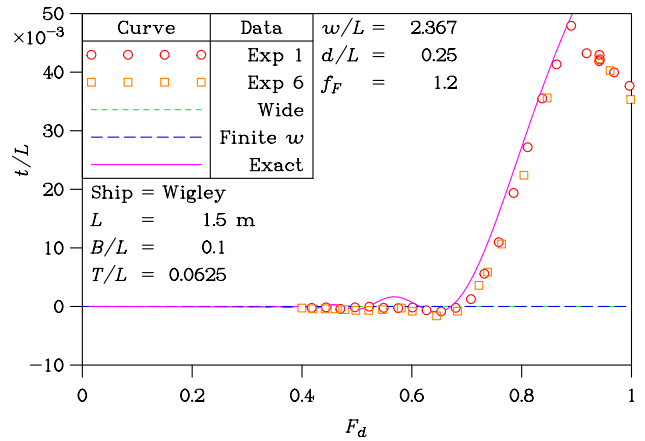
**Figure 3: Sinkage**  
(a)  $d/L = 0.1$



**Figure 3: Sinkage**  
(b)  $d/L = 0.25$



**Figure 4: Trim**  
(a)  $d/L = 0.1$



**Figure 4: Trim**  
(b)  $d/L = 0.25$

the theory approaches the expected limit of the *hydraulic theory* (Curve 1) for a narrow channel  $w/L = 0.5$ . Similarly, the desired limit of the *wide-channel theory* (Curve 6) of Tuck (1966) is closely achieved for the case of  $w/L = 4$ .

A comparison of the predicted and experimental sinkage is presented in Figure 3(a) for a relatively low depth-to-length ratio  $d/L = 0.1$ . Finite-width shallow-water theory represents an improvement over the wide-shallow-water theory.

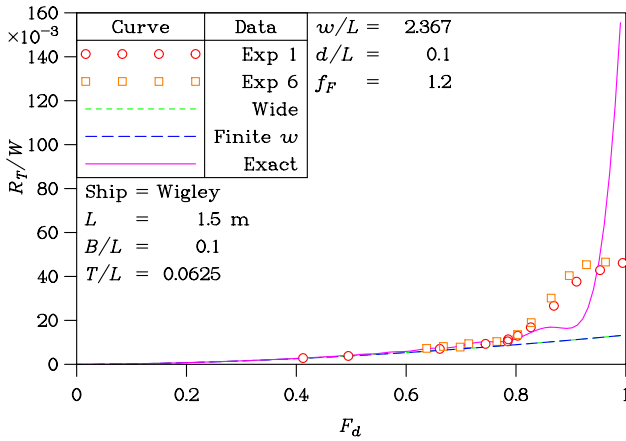
Both theories are a good indication of the sinkage. Nevertheless, the more traditional finite-width finite-depth (exact theory) is better still. The case of deeper water, namely  $d/L = 0.25$ , in Figure 3(b) shows that the shallow-water concept is much less valid in this case.

## 5 Trim

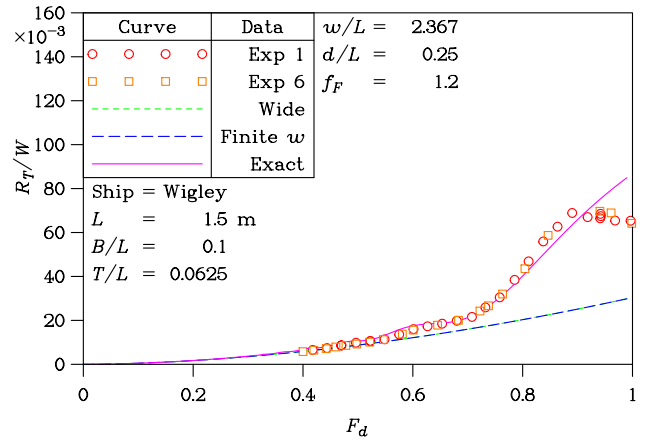
The trim is depicted in the two parts of Figure 4 for the same two water depths. The shallower case in Figure 4(a) is particularly interesting, because both shallow-water theories indicate no trim, for these subcritical speeds, for this vessel with fore-aft symmetry.

This is indeed reasonably true in practice, at least up to  $F_d \approx 0.8$ . The exact theory does provide a positive trim at higher speeds, in keeping with observation.

At the greater depth of  $d/L = 0.25$  in Figure 4(b), the trim is certainly nonzero for  $F_d > 0.7$ ; the exact theory provides a good prediction for values of  $F_d$  up to about 0.9.



**Figure 5: Total Resistance**  
(a)  $d/L = 0.1$



**Figure 5: Total Resistance**  
(b)  $d/L = 0.25$

## 6 Resistance

Finally, the resistance for the lesser of the two water depths is shown in Figure 5(a). Curiously, the shallow-water theory predicts no wave resistance, so the only contribution to the total resistance  $R_T$  is through the frictional resistance, which was computed using the 1957 International Towing Tank Committee (ITTC) formulation, together with a frictional form factor  $f_F = 1.2$ .

The resistance is rendered dimensionless by the vessel weight  $W$ . The exact formulation provides a nonzero wave resistance in Figure 5(a). For the greater depth in Figure 5(b), the “exact” theory gives a good indication of the total resistance at speeds corresponding to almost  $F_d = 0.9$ .

## 7 Concluding Comments

The lowest depth corresponded to a depth-to-draft ratio  $d/T$  of 1.6. It would be instructive to conduct experiments at lower depths, where the shallow-water theory will be more applicable.

This study has been confined to the subcritical-speeds. The nature of the theoretical formulations is substantially different for the supercritical case — another fruitful area of research.

The tests were performed in the Towing Tank at the Australian Maritime College (AMC) under the supervision of Mr Gregor Macfarlane.

## 8 References

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