Evaluation of Time-Domain Capillary-Gravity Green Function

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We present this paper, in memory of Professor Fritz Ursell, which contains an exercise following his work on the generalization of steepest descent method in 1960, 1964 and 1968. The potential flow generated by an impulsive point source at the free surface with surface tension is analyzed and different asymptotic expressions have been obtained. They include a series expansion for small time, asymptotic expansion for very large time and uniform asymptotic expansions including Airy function and its derivative for intermediate and large time.

1 Introduction

The surface tension and fluid viscosity are neglected in the classical potential theory. Green function representing the velocity potential due to an impulsive disturbance presents a perplexing peculiarity - the surface elevation in a region approaching to the disturbance is found to oscillate with indefinitely increasing amplitude and indefinitely decreasing wavelength as pointed out in Lamb (1932), Ursell (1960), Clement (1998) and Chen & Wu (2001). The investigation of Chen (2002) and Chen & Duan (2003) shows that the introduction of surface tension in the formulation of ship waves eliminates the singularity of ship waves in the region near the track of the source point at the free surface. This stimulates our study on numerical evaluation of time-domain capillary-gravity Green function.

We study the potential at the point $P(x, y, z)$ and time instant t', generated at the point $Q(\xi, \eta, \zeta)$ and time t by a source of unit impulsive strength $\delta(t)$. The time-domain capillary-gravity Green function is the sum of an impulsive term and a memory part. The memory part is given by Wehausen & Laitone (1964, eq.24.28) and expressed by a wavenumber integral :

$$G(P, t', Q, t) = 2\int_0^\infty e^{k(z+\zeta)}\mathbf{J}_0(kR)\sqrt{gk + (T/\rho)k^3}\sin[\sqrt{gk + (T/\rho)k^3}(t - t')]dk \tag{1}$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ is the horizontal distance between the two points P and Q, g is the acceleration due to gravity, ρ water density and T surface tension on the air-water interface. $\mathbf{J}_0(\cdot)$ is the zeroth-order Bessel function of the first kind. If we use L as a reference length to write the non-dimensional quantities as

$$\tau = (t - t')\sqrt{g/L}; \quad (c, h) = (z + \zeta, R)/L; \quad \sigma = \sqrt{T/(\rho g L^2)}$$

the memory part G is written as

$$G(P, t', Q, t) = 2\sqrt{g/L^3}F(c, h, \tau) \tag{2}$$

such that the Green function is written in its dimensionless form :

$$F(c, h, \tau) = \int_0^\infty e^{kc}\mathbf{J}_0(kh)\omega(k)\sin[\omega(k)\tau]dk \quad \text{with} \quad \omega(k) = \sqrt{k + \sigma^2 k^3} \tag{3}$$

If we take $\sigma = 0$, the expression (3) keeps the same form as that of pure-gravity waves.

2 Contour integrals passing through the saddle points

By using the identity $\mathbf{J}_0(kh) = \mathbf{H}_0^+(kh) + \mathbf{H}_0^-(kh)$ with the Hankel function $\mathbf{H}_0^\pm(kh) = \mathbf{J}_0(kh) \pm i\mathbf{Y}_0(kh)$ in (3), we may decompose $F = F^+ + F^-$ with F^+ and F^- associated with $\mathbf{H}_0^+$ and $\mathbf{H}_0^-$, respectively. At large values of $\omega(k)\tau$ and kh, the integrand of F^+ and F^- is of highly oscillatory. The phase function in the integrand of F^+ is $\psi^+ = \omega(k)\tau + kh$ while that of F^- is $\psi^- = \omega(k)\tau - kh$, identified by using the asymptotic expression of $\mathbf{H}_0^\pm(kh)$ for large kh.

The phase function $\psi^+ = \omega(k)\tau + kh$ does not present any saddle point for $\Re\{k\} > 0$. The integration path along the real axis of k for F^+ can then be deformed to the path along the imaginary axis of k since the arc integral linking both axis at infinity can be shown to be nil.

Following the work by Chen & Duan (2003) there are two saddle points k_g and k_T associated with the phase function $\psi^- = \omega(k)\tau - kh$ in the integrand of F^- :

$$k_g = 1/(4v^2) + O(\sigma/v^2) \quad \text{and} \quad k_T = 4v^2/(9\sigma^2) + O(\sigma/v^2) \quad \text{for} \quad v \gg \sqrt{\sigma} \tag{4}$$

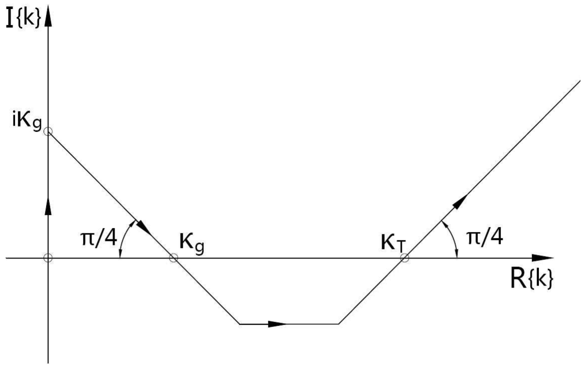

Figure 1: Integral contour for $v > v_0$

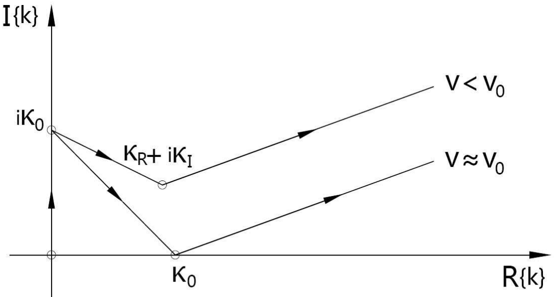

Figure 2: Integral contour for $v \leq v_0$

where $v = h/\tau$ is the wave velocity. When v is of the same order as $\sqrt{\sigma}$, the wavenumbers k_g and k_T become close and in particular, $k_g = k_T = k_0 \approx 0.393/\sigma$ for $v = v_0 \approx 1.086\sqrt{\sigma}$. When $v < v_0$, the wavenumbers k_g and k_T are complex. At the limit $v = 0$, we have $k_g = -i0.577/\sigma = -k_T$.

For $v > v_0$, the double derivatives $\partial^2\psi^-/\partial k^2(k = k_g) < 0$ and $\partial^2\psi^-/\partial k^2(k = k_T) > 0$ so that the steepest paths through $k = k_g$ and $k = k_T$ should be those with increments $\Delta k = |\Delta k|e^{-i\pi/4}$ and $\Delta k = |\Delta k|e^{+i\pi/4}$ assuming $|c| \ll 1$, respectively. At very large $k > k_T$, any path in the first quadrant is good although the best is that with increment $\Delta k = |\Delta k|e^{+i\pi/3}$ for $k \to \infty$. Furthermore, the contribution from the end point at $k = 0$ is taken account by an integration on a limited segment along the imaginary axis. The integration paths are depicted on Figure 1 and Figure 2 for $v > v_0$ and $v \leq v_0$, respectively. The Green function computed along the paths defined above is used to compare the asymptotic expansions in the following.

3 Series expansions for small τ

By expanding the sine function and $\sqrt{k + \sigma^2 k^3}$ for small τ and σ, we obtain

$$F(c, h, \tau) = \int_0^\infty e^{kc}\mathbf{J}_0(kh)\sqrt{k + \sigma^2 k^3}\sin(\tau\sqrt{k + \sigma^2 k^3})dk$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\frac{(-1)^m\tau^{2m+1}}{(2m+1)!}\frac{(m+1)!}{n!(m+1-n)!}\sigma^{2m+2-2n}\int_0^\infty e^{kc}\mathbf{J}_0(kh)k^{3m+3-2n}dk$$

Using the identity (6.621) in Gradshteyn & Ryzhik (2007) :

$$\int_0^\infty e^{kc}\mathbf{J}_0(kh)k^{3m+3-2n}dk = (3m+3-2n)!P_{3m+3-2n}(\cos\theta)\, r^{-(3m+4-2n)} \tag{5}$$

where

$$r = \sqrt{h^2 + c^2} \quad \text{and} \quad \cos\theta = -c/r$$

we find that $F(c, h, \tau)$ is expressed in the form

$$F(c, h, \tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\frac{(-1)^m\tau^{2m+1}}{(2m+1)!}\frac{(m+1)!(3m+3-2n)!}{n!(m+1-n)!}\sigma^{2m+2-2n}P_{3m+3-2n}(\cos\theta)\, r^{-(3m+4-2n)} \tag{6}$$

which can be used for small τ.

4 Expansions for very large τ

At very large time of τ, the behaviors of $F(c, h, \tau)$ depend on the contributions of integration in the vicinity of end points and saddle points. To estimate the contribution from the end points, we change the integral variable k to ω and rewrite $F(c, h, \tau)$ as

$$F(c, h, \tau) = 2\int_0^\infty e^{kc}\mathbf{J}_0(kh)\omega^2/(1 + 3\sigma^2 k^2)\sin(\omega\tau)d\omega \quad \text{with} \quad \varphi = \psi^-/\tau = \omega(k) - kv \tag{7}$$

in which the amplitude function of the integrand is expanded at $\omega \to 0$ as

$$e^{kc}\mathbf{J}_0(kh)\omega^2/(1+3\sigma^2k^2) = \omega^2 + c\omega^4 + (-3\sigma^2 - h^2/4 + c^2/2)\omega^6 + O(\omega^8) \tag{8}$$

Introducing (8) into (7) and using the identity

$$\int_0^\infty \omega^n \sin(\omega\tau)d\omega = \cos(n\pi/2)n!/\tau^{n+1} \tag{9}$$

we have

$$F(c,h,\tau) \approx -4/\tau^3 + 48c/\tau^5 + 5!(36\sigma^2 + 3h^2 - 6c^2)/\tau^7 + O(\tau^{-9}) \tag{10}$$

The expression (10) representing the contribution from the end point at $k = 0$ for very large τ should be added to the contributions from saddle points which are now developed below.

5 Uniform asymptotic expansions

For the large value of h and τ, the dominant contribution comes from the saddle points in the integrand of $F^-(c,h,\tau)$ and we write (3) in the complex form :

$$F(c,h,\tau) \approx F^- = \Im\left\{\int_0^\infty \frac{\omega e^{kc+\pi i/4}}{\sqrt{2\pi kh}} e^{i\tau\varphi(k,v)}dk\right\} \quad \text{with} \quad \tau\varphi(k,v) = \psi^-(\tau,h,k) \tag{11}$$

To develop an uniform asymptotic expansion, we use the method of Chester, Friedman & Ursell (1957) and define a cubic transform of the variable of integration k to u

$$i\varphi(k,v) = i(\omega - kv) = -(u^3/3 - \gamma^2 u) + \rho \tag{12}$$

it follows that

$$F(c,h,\tau) \approx \Im\left\{e^{\tau\rho}\int_{\infty e^{2\pi i/3}}^{\infty e^{4\pi i/3}} G_0(u,v)e^{-\tau(u^3/3-\gamma^2 u)}du\right\} \tag{13}$$

where

$$G_0(u,v) = \sqrt{(1+\sigma^2k^2)/(2\pi h)}e^{kc+\pi i/4}(dk/du) \tag{14}$$

We then construct a Bleistein sequence to replace the integrand $G_0(u,v)$

$$G_0(u,v) = b_0 + b_1 u + (u^2 - \gamma^2)H_0(u,v) \tag{15}$$

We obtain the uniform asymptotic expansions of $F(c,h,\tau)$ for large τ

$$F(c,h,\tau) \approx \Im\left\{2\pi i e^{\tau\rho}[\mathbf{Ai}(\tau^{2/3}\gamma^2)b_0/\tau^{1/3} + \mathbf{Ai}'(\tau^{2/3}\gamma^2)b_1/\tau^{2/3}]\right\} \tag{16}$$

where

$$\rho = i(\varphi(k_T,v) + \varphi(k_g,v))/2 \tag{17}$$

$$\gamma = \begin{cases} \exp(\pi i/2)\{[\varphi(k_g,v) - \varphi(k_T,v)]3/4\}^{1/3} & \text{for} \quad v > v_0 \\ 0 & \text{for} \quad v = v_0 \\ \exp(\pi i)\{\Im[\varphi(k_T,v) - \varphi(k_g,v)]3/4\}^{1/3} & \text{for} \quad v < v_0 \end{cases} \tag{18}$$

and

$$b_0 = \begin{cases} e^{i\pi/4}/\sqrt{8\pi h}\left(\sqrt{1+\sigma^2k_g^2}e^{k_g c}\ (dk/du)|_{u=-\gamma} + \sqrt{1+\sigma^2k_T^2}e^{k_T c}\ (dk/du)|_{u=\gamma}\right) & \text{for} \quad v \neq v_0 \\ e^{-i\pi/4}/\sqrt{2\pi h}\sqrt{1+\sigma^2k_0^2}\ (2/\varphi'''_{kkk}(k_0,v))^{1/3} & \text{for} \quad v = v_0 \end{cases} \tag{19}$$

$$b_1 = \begin{cases} e^{i\pi/4}/(\gamma\sqrt{8\pi h})\left(\sqrt{1+\sigma^2k_T^2}e^{k_T c}\ (dk/du)|_{u=\gamma} - \sqrt{1+\sigma^2k_g^2}e^{k_g c}\ (dk/du)|_{u=-\gamma}\right) & \text{for} \quad v \neq v_0 \\ 0 & \text{for} \quad v = v_0 \end{cases} \tag{20}$$

with

$$\left.\frac{dk}{du}\right|_{u=\mp\gamma} = \begin{cases} e^{-i\pi/2}\sqrt{2\,|\gamma/\varphi''_{kk}(k_{g,T},v)|} & \text{for} \quad v > v_0 \\ e^{-i[\pi\pm\arg(i/\varphi''_{kk}(k_T,v))]/2}\sqrt{2\,|\gamma/\varphi''_{kk}(k_{g,T},v)|} & \text{for} \quad v < v_0 \end{cases} \tag{21}$$

The resulting expansion (16) is uniformly valid for a large zone $|v - v_0| < M_v$ independent of τ and more details can be found in Dai and Chen (2012). The pure gravity waves on finite depth due to an impulse have the similar situation and studied in Clarisse *et al.* (1995).

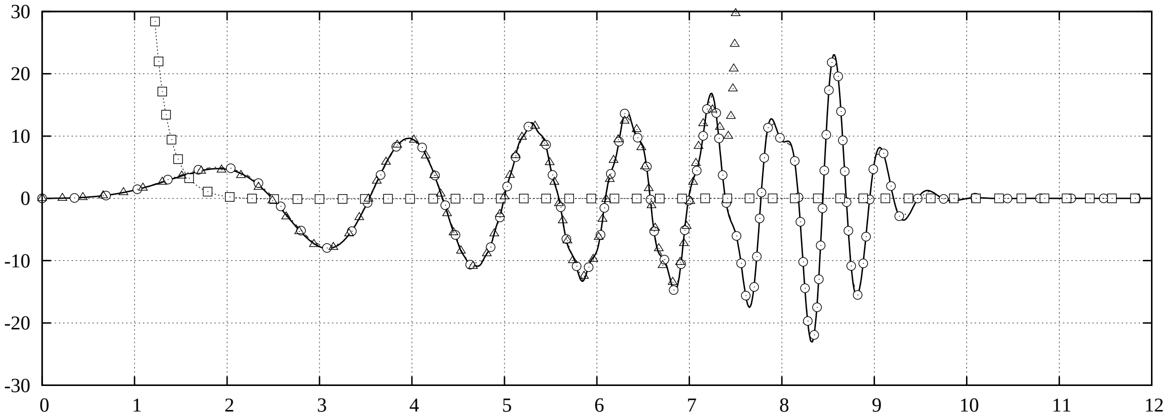

Figure 3: Comparison of asymptotic expansions with contour integrals for $(c, h) = (-0.01, 0.5)$

6 Summary and conclusions

Based on the above study, $F(c, h, \tau)$ can be evaluated by the contour integration along the steepest descent paths passing through the saddle points. The results by contour integrals are useful to check the asymptotic expansions. An example of numerical results for τ varying from 0 to 12 at (c, h)=$(-0.01, 0.5)$ is depicted on Figure 3. The results of contour integral are represented by solid lines while those of the uniform asymptotic expansions (16) by the symbol of empty circles. The results of series expansions (6) are depicted by the symbol of empty triangles and those of (10) by the symbol of empty squares. For small and moderate values of τ, the series expansion (6) gives excellent results. For large values of τ the asymptotic expansions (10) representing the contribution from the end point at $k = 0$ and the uniform asymptotic expansion (16) corresponding to the contribution of saddles points at $k = k_g$ and $k = k_T$ are very good as well. Finally, at very large values of τ, the uniform asymptotic expansion (16) is of exponentially small $O(e^{-\tau|\gamma|^3 2/3})$ so that asymptotic expansions (10) becomes dominant theoretically but the absolute value remains very small.

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