A second order Ordinary Differential Equation for the frequency domain Green function.

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The present work is developped under the usual assumptions of linear potential flow theory. We consider a source point $Q'(x', y', z')$, and a field point $Q(x, y, z)$ both lying in the lower half-plane $(z < 0, z' < 0)$. r is the relative horizontal distance, $Z = z + z'$ the vertical distance between the field point and $Q'_1$the image of the source point relative to the free surface (fig.1). R is the distance $|QQ'|$ and $R_1 = \sqrt{r^2 + Z^2}$ the distance $|QQ'_1|$.

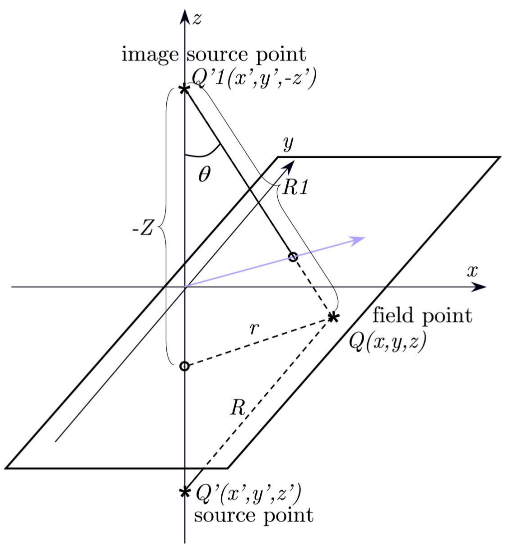

Figure 1: source and field point - layout and notations

1 Time and Frequency domain Green functions

Frequency-domain - Let us first recall the expression of the complex Green function of the diffraction-radiation problem in water of infinite depth in non-dimensionalized variables.

$$G_\infty(r, Z, i\omega) = \left(\frac{1}{R} - \frac{1}{R_1}\right) + 2\text{PV}\int_0^\infty \frac{k}{k - k_0} e^{kZ} J_0(kr)dk - 2i\pi sgn(\omega) k_0 e^{k_0 Z} J_0(k_0 r) \tag{1}$$

or

$$G_\infty(r, Z, i\omega) = G_0(Q, Q') + \ G(r, Z, i\omega)$$

where $k_0 = \omega^2$, and PV$\int$ means principal value of the integral.

This expression can be shown to be equivalent to Wehausen [6] formulation:

$$G_\infty(r, Z, i\omega) = \frac{1}{R} + \text{PV}\int_0^\infty \frac{k + k_0}{k - k_0} e^{kZ} J_0(kr)dk - 2i\pi sgn(\omega) k_0 e^{k_0 Z} J_0(k_0 r) \tag{2}$$

by using the identity :$(R_1)^{-1} = \left(r^2+Z^2\right)^{-\frac{1}{2}} = \int_0^\infty e^{kZ} J_0(kr)dk$ [4]. Eq (1) gives the velocity potential at point Q generated by a pulsating source of unit strenght located at point Q'. The G_∞ function (1) is the basic mathematical tool used to solve diffraction-radiation problems in regular waves of frequency ω, and period $T = 2\pi/\omega$.

Time-domain - when the source strength is no longer a cosine function of time but a simple impulse, namely $q(t) = \delta(t)$ a Dirac function, then the Green function is given by

$$F_\infty(r,Z,t) = \left(\frac{1}{R} - \frac{1}{R_1}\right)\delta(t) + 2\mathbb{H}(t)\int_0^\infty \sqrt{k} e^{kZ} J_0(kr)\sin(t\sqrt{k})dk \tag{3}$$

or

$$F_\infty(r,Z,t) = G_0(Q,Q')\delta(t) + \mathbb{H}(t)F(r,Z,t)$$

with $\mathbb{H}(t)$ the Heaviside step function. Considering the velocity potential as the output of the fluid *system* with source strength as the input, it is established that the Green function of frequency domain eq.(1) can be derived as the Fourier transform of the time-domain Green function eq.(3), with the following convention and notations:

$$G_\infty(r,Z,i\omega) = \hat{F}_\infty(r,Z,i\omega) = \mathcal{F}\left\{F_\infty(r,Z,t)\right\} = \int_{-\infty}^{\infty} F_\infty(r,Z,t)e^{-i\omega t}dt \tag{4}$$

In [2] [1], we have shown the integral part $F(r,Z,t)$ of the above time-domain Green function to be the solution of an exact fourth order linear ordinary differential equation with polynomial coefficient,

$$\left(r^2+Z^2\right)F^{(4)} - ZtF^{(3)} + \left(\frac{1}{4}t^2 - 4Z\right)F^{(2)} + \frac{7}{4}tF^{(1)} + \frac{9}{4}F = 0 \tag{5}$$

where the notation $F^{(n)}$ holds for the n^{th} derivative of F with respect to the time variable t. Among other applications, this ODE has been shown [3] to provide a good mean to speed up the inline computations of the Green function in time-domain BEM computational codes.

In this paper, we will show how to derive from eq.5 a similar ODE, but in the frequency variable, for the frequency domain Green function $G(r,Z,i\omega)$.

2 Fourier transform of the time-domain ODE

Let us first introduce the auxiliary function $S(r,Z,t) = \mathbb{H}(t)F(r,Z,t)$ with $\mathbb{H}$ the Heaviside step function.

Taking into account the differential relation between the Heaviside and Dirac functions, i.e $\delta(t) = d\mathbb{H}(t)/dt = \mathbb{H}^{(1)}(t)$, we establish the following relations by successive differentiation

$$\begin{aligned}
S &= \mathbb{H}F \\
S^{(1)} &= \delta F + \mathbb{H}F^{(1)} \\
S^{(2)} &= \delta^{(1)}F + 2\delta F^{(1)} + \mathbb{H}F^{(2)} \\
S^{(3)} &= \delta^{(2)}F + 3\delta^{(1)}F^{(1)} + 3\delta F^{(2)} + \mathbb{H}F^{(3)} \\
S^{(4)} &= \delta^{(3)}F + 4\delta^{(2)}F^{(1)} + 6\delta^{(1)}F^{(2)} + 4\delta F^{(3)} + \mathbb{H}F^{(4)}
\end{aligned}$$

The time domain homogeneous ODE (5) obviously still holds when both sides are multiplied by $\mathbb{H}(t)$, leading to:

$$\begin{aligned}
\left(r^2+Z^2\right)S^{(4)} - ZtS^{(3)} + \left(\frac{1}{4}t^2 - 4Z\right)S^{(2)} + \frac{7}{4}tS^{(1)} + \frac{9}{4}S = \\
+\delta\left[\left(r^2+Z^2\right)F^{(3)} - 3ZtF^{(2)} + 2\left(\tfrac{1}{4}t^2 - 4Z\right)F^{(1)} + \tfrac{7}{4}tF\right] \\
+\delta^{(1)}\left[6\left(r^2+Z^2\right)F^{(2)} - 3ZtF^{(1)} + \left(\tfrac{1}{4}t^2 - 4Z\right)F\right] \\
+\delta^{(2)}\left[4\left(r^2+Z^2\right)F - ZtF\right] \\
+\delta^{(3)}\left(r^2+Z^2\right)F
\end{aligned} \tag{6}$$

2.1 Left-hand side

As expected, the left hanfd side of (6) is the same as eq.5. Let's now take its Fourier transform

$$\mathcal{L} = \mathcal{F}\left\{\left(r^2+Z^2\right)S^{(4)} - ZtS^{(3)} + \left(\frac{1}{4}t^2 - 4Z\right)S^{(2)} + \frac{7}{4}tS^{(1)} + \frac{9}{4}S\right\} \tag{7}$$

By using the basic rules of Fourier transform for a derivative $F\left\{f^{(n)}(t)\right\} = (i\omega)^n \hat{f}(i\omega)$, and for the product by a polynomial $F\left\{t^m f(t)\right\} = i^m \frac{d^m}{d\omega^m}\hat{f}(i\omega)$ [5], and after re-ordering we get:

$$\mathcal{L} = \frac{\omega^2}{4}\hat{S}^{(2)} - \omega\left(\omega^2 Z + \frac{3}{4}\right)\hat{S}^{(1)} + \left(\omega^4\left(r^2+Z^2\right) + \omega^2 Z + 1\right)\hat{S} \tag{8}$$

2.2 Right-hand side

For developping the Fourier transform of the right hand side of (6), we then apply the following relations derived from the fundamental property of the Dirac delta function, and integration by parts:

$$\begin{aligned}
\mathcal{F}\left\{\delta(t)f(t)\right\} &= \int_{-\infty}^{+\infty}\delta(t)f(t)e^{-i\omega t} = f(0) && (9)\\
\mathcal{F}\left\{\delta^{(1)}(t)f(t)\right\} &= \int_{-\infty}^{+\infty}\delta^{(1)}(t)f(t)e^{-i\omega t} = -f^{(1)}(0) + i\omega f(0)\\
\mathcal{F}\left\{\delta^{(2)}(t)f(t)\right\} &= \int_{-\infty}^{+\infty}\delta^{(2)}(t)f(t)e^{-i\omega t} = f^{(2)}(0) - 2i\omega f^{(1)}(0) - \omega^2 f(0)\\
\mathcal{F}\left\{\delta^{(3)}(t)f(t)\right\} &= \int_{-\infty}^{+\infty}\delta^{(3)}(t)f(t)e^{-i\omega t} = -f^{(3)}(0) + 3i\omega f^{(2)}(0) + 3\omega^2 f^{(1)}(0) - i\omega^3 f(0)
\end{aligned}$$

Then, after simple calculations and simplifications, the Fourier transform of the right-hand side $\mathcal{R}$, is found to be expressed simply as:

$$\mathcal{R} = \frac{2\left(1+Z\omega^2\right)}{\sqrt{(r^2+Z^2)}} \tag{10}$$

2.3 The complete frequency-domain ODE

From (1) and (3) is is clear that $\hat{S}(r,Z,i\omega)$ is nothing but the frequency dependent part G of $G_\infty(r,Z,i\omega)$. The ODE for $G(r,Z,i\omega)$ can thus be formed now from its left hand and right hand sides, derived above, giving:

$$\boxed{\frac{\omega^2}{4}G^{(2)} - \omega\left(\omega^2 Z + \frac{3}{4}\right)G^{(1)} + \left(\omega^4\left(r^2+Z^2\right) + \omega^2 Z + 1\right)G = \frac{2\left(1+Z\omega^2\right)}{\sqrt{(r^2+Z^2)}}} \tag{11}$$

So, for the frequency domain complex Green function $G(r,Z,i\omega)$, the ODE is of order 2, logically, due to the maximum order 2 of the polynomial coefficients of the time domain ODE (5). Similar 2nd order ODE can be derived for the gradient $\frac{\partial G}{\partial r}$ and $\frac{\partial G}{\partial Z}$ by applying straightforwardly the same method to the time domain ODE derived in [1] respectively for the gradient $\frac{\partial F}{\partial r}$and $\frac{\partial F}{\partial Z}$of the time domain Green function.

The initial conditions for the ODE eq.11 can be easily deduced from the developped expression of G in (1), leading to:

$$\left\{\begin{array}{c} G(r,Z,0) = \frac{2}{\sqrt{(r^2+Z^2)}} - i(0) \\ \lim\limits_{\omega\rightarrow 0}\left\{\frac{\partial}{\partial\omega}G(r,Z,i\omega)\right\} = 0 + i(0) \end{array}\right. \tag{12}$$

The results of a numerical check of this ODE (11) are shown on fig:2 and fig:3 were we have integrated it by a standard Runge-Kutta algorithm from $\omega = 0$ to 4. (test case parmeters: r=2, Z=-1). Here all

3 terms of the differential equation are plotted independently, together with their sum, real part on the left, imaginary part on the right. The isolated dots show the same quantities (real and imaginary parts of the LHS of eq:11) calculated from the usual Green function routines of the BEM code AQUAPLUS [8] based on the classical series and asymptotic expansions for this function [7]. An excellent agreement is observed for both real and imaginary parts.

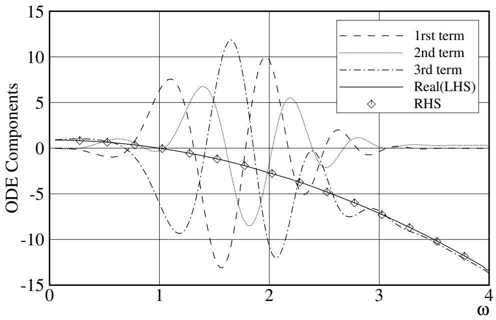

Figure 2: components of the ODE - Real part

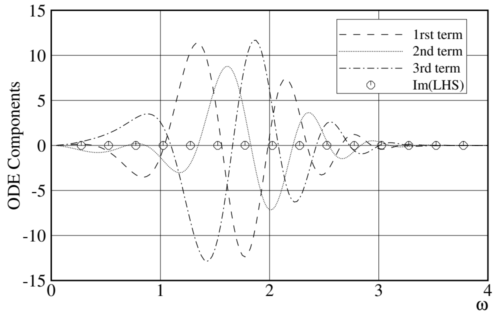

Figure 3: components of the ODE - imaginary part

3 conclusion

The Green functions of free surface hydrodynamics were shown to be solution of exact linear ordinary differential equations. This result, uncovered at the IWWWFB in Marseille (1997) for the time-domain Green function, is extended here for it's sister function in the frequency-domain. In both cases, theses results can be exploited to speed-up the computations of BEM based seakeeping codes. A lot of other applications will probably be discovered in further research works. The extension of these results to the case of finite water depth is still an open challenge.

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