

Hybrid-Spectral Model for Fully Nonlinear Numerical Wave Tank*

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Introduction

This abstract is concerned with the development of an efficient and scalable hybrid-spectral model for a fully nonlinear numerical wave tank with finite dimensions, which is used for simulation of nonlinear free surface waves generated by a moving wavemaker.

The approach pursued here is based on the OceanWave3D strategy established in [1, 6, 7], replacing the horizontal high order finite difference approximations used in OceanWave3D with a Fourier Collocation method based on cosine modes. This is combined with a modal Chebyshev Tau method for the potential in the vertical, which by applying an efficient strategy for solving the discrete Laplace equation yields a new attractive and accurate Dirichlet-to-Neumann operator, recently established in [4].

The application of hybrid-spectral methods in the context of the OceanWave3D strategy is motivated by the comparative study of two serial implementations of nonlinear water wave models carried out in [5], which demonstrated that the high-order finite difference model (OceanWave3D) requires approximately an order of magnitude larger computational effort than the high order spectral model (HOS) in order to solve highly nonlinear water wave problems to the same level of accuracy. Compared to the HOS model, the main advantage of OceanWave3D is a relatively straight-forward introduction of non-rectangular geometry, in particular varying bottom bathymetry and the generation of waves by a moving wavemaker. These attractive properties, associated with the OceanWave3D solution strategy, are retained in the presented hybrid-spectral method.

In this abstract the governing equations for the fully nonlinear free surface potential wave problem are derived in curvilinear coordinates on a fixed computational domain, which allows the fully nonlinear wavemaker condition to be satisfied directly. The hybrid-spectral discretization strategy and iterative solution of the resulting discrete Laplace problem are detailed, and it is described how the wavemaker can be modelled by the introduction of additional potentials following the line of [2], which with the present discretization strategy can be obtained essentially free of cost.

Problem Formulation

A Cartesian coordinate system $x_i = (x_1, x_2, x_3)$ is adopted with the x_1x_2 -plane located at the still water level and the x_3 -axis pointing vertically upwards. Indicical notation is invoked and the summation convention applies to repeated indices, with latin indices $i = 1, 2, 3$ accounting for all spatial dimensions, while greek indices $\alpha = 1, 2$ only apply to the horizontal dimensions. The still water depth is given by $x_3 = -h(x_\alpha)$ and the position of the free surface is defined by $x_3 = \zeta(x_\alpha, t)$. The wave tank is assumed rectangular in the horizontal plane with dimensions L_1, L_2 and vertical walls, and the gravitational acceleration g is assumed to be constant. Assuming an inviscid fluid and in irrotational flow, the fluid velocity $u_i = \partial_{x_i}\phi$ is defined by the gradient of a scalar velocity potential $\phi(x_i, t)$. The position of the free surface in an Eulerian frame of reference is captured by the usual kinematic boundary condition, while the dynamic free surface boundary condition follows from Bernoulli's equation,

$$\partial_t \zeta = \partial_{x_3} \phi - \partial_{x_\alpha} \phi \partial_{x_\alpha} \zeta, \quad (1a)$$

$$\partial_t \phi = -g\zeta - \frac{1}{2} \partial_{x_i} \phi \partial_{x_i} \phi. \quad (1b)$$

To evolve these equations in time requires solving the Laplace equation for ϕ in the fluid volume Ω . A well-posed Laplace problem is achieved by specifying a known ϕ at the free surface ζ , together with kinematic boundary conditions at the solid boundaries to the domain, i.e. the wave maker, walls and seabed,

$$\partial_{x_i x_i} \phi = 0, \quad x_i \in \Omega, \quad (2a)$$

$$n_i (\partial_{x_i} \phi - v_i^{solid}) = 0, \quad x_i \in \partial\Omega^{solid}, \quad (2b)$$

The free surface is a time-dependent moving boundary with an a priori unknown position, practically resulting in a time-dependent domain which may be handled efficiently using the well-known σ -transformation in the vertical [1]. Further introducing a wavemaker as a moving solid boundary, with an a priori known position $x_1 = F(x_2, x_3, t)$ provided by a paddle signal, yields the requirement for a more general mapping between the physical domain Ω and a fixed, time-independent computational domain Ω_C ,

$$x_i = x_i(\xi^j, t) \Leftrightarrow \xi^j = \xi^j(x_i, t), \quad \xi^j \in \Omega_C \quad (3)$$

where $\xi^j = (\xi^1, \xi^2, \xi^3)$ is a set of general curvilinear coordinates. The mapping $x_i = x_i(\xi^j, t)$ from computational to physical space is done using transfinite interpolation with linear blending of the boundary

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faces, i.e. the free surface, seabed, wavemaker and the fixed walls of the domain [8], while the inverse mapping $\xi^j = \xi^j(x_i, t)$ is not established explicitly though it formally indeed does exist. Taking the derivatives of the mappings $\partial_{\xi^j} x_j$ and $\partial_{x_i} \xi^j$ yield the Cartesian components of the covariant and contravariant basis vectors \mathbf{g}_j and \mathbf{g}^j , which also defines the co- and contravariant metric tensors g_{ij} and g^{ij} ,

$$(\mathbf{g}_j)_i = \partial_{\xi^j} x_i, \quad (\mathbf{g}^j)_i = \partial_{x_i} \xi^j, \quad (4a)$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j. \quad (4b)$$

Via the chain rule, these co- and contravariant basis vectors provide the relationship between spatial derivatives in the physical and computational spaces. For example, derivatives of ζ and ϕ with respect to x_i and ξ^j are related by

$$\partial_{x_\alpha} \zeta = \partial_{x_\alpha} \xi^j \partial_{\xi^j} \zeta = (\mathbf{g}^j)_\alpha \partial_{\xi^j} \zeta, \quad (5a)$$

$$\partial_{x_i} \phi = \partial_{x_i} \xi^j \partial_{\xi^j} \phi = (\mathbf{g}^j)_i \partial_{\xi^j} \phi. \quad (5b)$$

Note that $\partial_{x_3} \zeta = 0$ and hence only $\partial_{x_\alpha} \zeta$ are evaluated. Similarly the temporal derivatives of ζ, ϕ in physical and computational spaces are related as follows, due to [8],

$$(\partial_t \zeta)_{x_i} = (\partial_t \zeta)_{\xi^j} - (\partial_t x_\alpha)_{\xi^j} \partial_{x_\alpha} \zeta \quad (6a)$$

$$= (\partial_t \zeta)_{\xi^j} - w_\alpha (\mathbf{g}^j)_\alpha \partial_{\xi^j} \zeta,$$

$$(\partial_t \phi)_{x_i} = ((\partial_t \phi)_{\xi^j} - (\partial_t x_i)_{\xi^j} \partial_{x_i} \phi) \quad (6b)$$

$$= (\partial_t \phi)_{\xi^j} - w_i (\mathbf{g}^j)_i \partial_{\xi^j} \phi,$$

where $(\)_{\xi^j}, (\)_{x_i}$ indicates the spatial coordinates being kept fixed under partial time differentiation, while $w_i = (\partial_t x_i)_{\xi^j}$ denotes the transport velocity of the transient physical reference frame due to the motion of the boundaries.

The covariant basis vectors \mathbf{g}_j are immediately evaluated from the mapping function $x_i = x_i(\xi^j, t)$ at any instance in time by application of the differential operators ∂_{ξ^i} in computational space. From these the contravariant basis vectors \mathbf{g}^j , which are required to evaluate (5)-(6), are computed by cross products of \mathbf{g}_j as,

$$\mathbf{g}^k = (\sqrt{g})^{-1} \varepsilon_{ijk} \mathbf{g}_i \times \mathbf{g}_j, \quad \sqrt{g} = \sqrt{\det(g_{ij})}. \quad (7)$$

in which \sqrt{g} has been introduced as the Jacobian of the mapping and ε_{ijk} is the permutation operator. Similarly the contravariant metric tensor g^{ij} follows from the covariant tensor g_{ij} as its inverse,

$$g^{ij} g_{ij} = \delta_i^j \Rightarrow (g^{ij}) = (g_{ij})^{-1}, \quad (8)$$

where δ_i^j denotes the Kronecker delta.

Inserting (5)-(6), the free surface evolution equa-

tions (1) may be cast in computational space as,

$$\partial_t \zeta = w_\alpha (\mathbf{g}^j)_\alpha \partial_{\xi^j} \zeta + (\mathbf{g}^j)_3 \partial_{\xi^j} \phi \quad (9a)$$

$$- ((\mathbf{g}^j)_\alpha \partial_{\xi^j} \phi) ((\mathbf{g}^k)_\alpha \partial_{\xi^k} \zeta),$$

$$= (\mathbf{g}^j)_3 \partial_{\xi^j} \phi - ((\mathbf{g}^j)_\alpha \partial_{\xi^j} \phi - w_\alpha) ((\mathbf{g}^k)_\alpha \partial_{\xi^k} \zeta),$$

$$\partial_t \phi = w_i (\mathbf{g}^j)_i \partial_{\xi^j} \phi - g \zeta \quad (9b)$$

$$- \frac{1}{2} ((\mathbf{g}^j)_i \partial_{\xi^j} \phi) ((\mathbf{g}^k)_i \partial_{\xi^k} \phi),$$

$$= -g \zeta - \frac{1}{2} ((\mathbf{g}^j)_i \partial_{\xi^j} \phi - w_i) ((\mathbf{g}^k)_i \partial_{\xi^k} \phi),$$

which shows that the transient physical domain is captured by recasting the free surface equations in arbitrary Eulerian-Lagrangian form in computational space. As a remark the widely used Zakharow form of the free surface equations, applied e.g. in [1], may be derived from (9) simply by taking the mapping $x_i = x_i(\xi^j, t)$ to be the inverse of the σ -transform.

The Laplace equation and the kinematic solid boundary condition (2) are similarly expressed in curvilinear coordinates as follows, due to [8],

$$(\sqrt{g})^{-1} \partial_{\xi^i} (\sqrt{g} g^{ij} \partial_{\xi^j} \phi) = 0 \quad \xi^j \in \Omega_C, \quad (10a)$$

$$n_i ((\mathbf{g}^j)_i \partial_{\xi^j} \phi - v_i^{solid}) = 0, \quad \xi^j \in \partial \Omega_C^{solid}, \quad (10b)$$

in which it is assumed that the normal vector n_i to the physical solid boundaries and the solid boundary velocity v_i^{solid} are expressed in computational space.

Numerical Methods

A method of lines approach is adopted for the discretization of the governing equations stated above. For the time-integration of the free-surface equations (9), the classic four-stage, fourth-order Runge-Kutta scheme is employed as it is not subject to any severe stability constraint on the choice of time steps.

The governing equations are discretized in the computational space using a hybrid-spectral collocation method detailed in [4], combining a nodal Fourier collocation method on a horizontal grid of $\mathbf{M} = (N_1, N_2)$ grid points with a modal Chebyshev Tau method in the vertical truncated to the first $N_3 = N + 1$ Chebyshev polynomials, where N is the polynomial order. Due to the finite dimensions and rectangular shape of the numerical wave tank (with the wavemaker at neutral position), it was suggested by [2] to employ a tensor product of cosine modes, i.e. real-and-even Fourier modes, as the basis for the horizontal Fourier collocation method, since they are the natural modes of the tank. Hence ζ, ϕ have the global representation in the computational domain,

$$\zeta(\xi^\alpha, t) = \sum_{m=0}^{M-1} \zeta_m(t) l_m(\xi^\alpha), \quad (11a)$$

$$\phi(\xi^j, t) = \sum_{m=0}^{M-1} \sum_{n=0}^N \hat{\phi}_{mn}(t) l_m(\xi^\alpha) T_n(\xi^3), \quad (11b)$$

where $\zeta_m(t) = \zeta(\boldsymbol{\xi}_m, t)$ and $\hat{\phi}_{mn}(t) = \hat{\phi}_n(\boldsymbol{\xi}_m, t)$ are the solutions at the collocation nodes $\boldsymbol{\xi}_m =$

$(m + \frac{1}{2})/M\pi$, while $T_n(\xi^3) = \cos(n \arccos(\xi^3))$ are the Chebyshev polynomials and $l_m(\xi^\alpha)$ are globally defined nodal Lagrange interpolation polynomials, which may be formally derived from the cosine modes $\psi_m(\xi^\alpha) = \cos(m_1\xi^1)\cos(m_2\xi^2)$. Finally $\mathbf{m} = (m_1, m_2)$ is a multi-index where the two components of \mathbf{m} are allowed to vary independently.

The horizontal derivatives ∂_{ξ^α} are approximated by their discrete counterparts in computational space using the Fast Fourier/Cosine Transform (FFT/FCT) to evaluate the derivatives in the collocation nodes. The Chebyshev Tau method provides an efficient, spectrally accurate Dirichlet-to-Neumann operator for the evaluation of the gradient of the velocity potential at the free surface. Based on orthogonal truncation rather than interpolation (as used e.g. in finite difference methods), the Chebyshev Tau method seeks to satisfy the Laplace equation (10a) over the depth for each vertical set of collocation nodes in weak form by requiring the residual to be orthogonal to a set of test functions $T_q(\xi^3)$ for $q = 0, \dots, N - 2$. The remaining two equations, required to obtain a square system of N_z equations with the N_z unknowns $\hat{\phi}_{mn}$ for each vertical set of collocation nodes, arise by imposing additional constraints on the coefficients in order to satisfy the free surface and bottom boundary conditions. For further details on the Chebyshev Tau method see e.g. [3].

Applied to the Laplace equation (10a) the Chebyshev Tau method yields a set of convolution sums, which for relatively simple mappings between the physical and the computational domain, such as the σ -transform, may be evaluated efficiently in modal space. The costs associated with the evaluation of the Laplacian in such cases by direct convolution sums are comparable to those of applying a sixth order finite difference method. However, as demonstrated in [4] the Chebyshev Tau method introduces significantly lower dispersion errors than comparable finite difference methods. For more general mappings it may be advantageous to evaluate the convolution sums by pseudo-spectral products in the corresponding nodal space through application of the FCT, which applies to Chebyshev polynomials.

The resulting discrete Laplace problem can be stated as a rank $n = N_1N_2N_3$ linear system of equations $\mathcal{A}\Phi = b$, where \mathcal{A} is a large dense, non-symmetric matrix (in case the FFTs are replaced with matrix-based discrete Fourier transforms), Φ is a vector of values for the unknown scalar velocity potential, while b is a vector accounting for the inhomogeneous boundary conditions. Since the discrete Laplace problem is one spatial dimension $\mathcal{O}(N_z)$ larger than the free surface problem, the solution of this linear system at every time step is consequently the computational bottleneck of the model.

The discrete Laplace problem can be solved efficiently (i.e. with optimal scaling of the computa-

tional effort and memory footprint) by a iterative left preconditioned defect correction (PDC) method, as detailed in [4]. Evaluation of the residual $r = b - \mathcal{A}\phi$ and hence the matrix-vector product $\mathcal{A}\phi$ does not require \mathcal{A} to be formed explicitly and may be evaluated in $\mathcal{O}(n \log N_1)$ and $\mathcal{O}(n \log N_2)$ operators by application of the FCT horizontally.

The action of the preconditioning problem in the PDC method is to compute the correction δ to the current solution by solving the linear system of equations $\mathcal{M}\delta = r$, where $\mathcal{M} \approx \mathcal{A}$ is the preconditioning matrix. For the present problem and hybrid-spectral discretization method, the corresponding constant coefficient Laplace problem (i.e. neglecting bottom variations, free surface elevation and wavemaker motion) provides an attractive efficient and sparse preconditioning strategy. In order to avoid assembling and factorizing the full dense preconditioning matrix, the residual r is first transformed to a modal cosine representation horizontally by use of the FCT, while the modal Chebyshev representation is retained in the vertical. This decouples the preconditioning step into a inhomogeneous constant coefficient Helmholtz equation in the vertical for each horizontal cosine mode, since $r \neq 0$ and $\nabla^2\psi_m(\xi^\alpha) = -\mathbf{m}\cdot\mathbf{m}\psi_m(\xi^\alpha)$.

Applying the Chebyshev Tau method in the vertical to each of the Helmholtz equations yields N_1N_2 sparse systems of N_3 equations at each horizontal collocation point. These systems are primarily quasi-pentadiagonal as each row contains only three non-zero entries in diagonals $(-2, 0, +2)$ in addition to two dense rows enforcing the boundary conditions, see [3]. Using a tailored Gaussian elimination solution strategy, each of these systems is solved in just $18N_3$ operations, only around twice the cost of the classic tridiagonal matrix algorithm, and with the same $2p$ memory footprint, making this preconditioning strategy well suited for parallel implementation.

Wave Generation by Additional Potentials

The wavemaker is modelled as a solid moving boundary with a known position $x_1 = F(x_2, x_3, t)$, which gives rise to the following free slip condition,

$$\partial_t F = \partial_{x_1}\phi - \partial_{x_2}F\partial_{x_2}\phi - \partial_{x_3}F\partial_{x_3}\phi = \partial_n\phi, \quad (12)$$

where ∂_n denotes the derivative normal to the wavemaker with normal vector $n_i^{gen} = (1, -\partial_{x_2}F, -\partial_{x_3}F)$. The velocity potential (11b) defined in terms of cosine modes implicitly satisfies homogeneous Neumann boundary conditions, hence it cannot account for the inhomogeneous Neumann boundary condition introduced by the wavemaker motion. This was also noted by [2], who suggested the use of additional potentials to account for the inhomogeneous Neumann condition by splitting the potential into two,

$$\phi = \phi_P + \phi_G, \quad (13)$$

where ϕ_P is the propagating spectral potential defined in (11b) and ϕ_G is the generation potential accounting for the inhomogeneous Neumann condition (12) at the wavemaker, while the total potential ϕ is advanced by the free surface equations (9).

In this work we seek to establish the generation potential ϕ_G by exploiting that the Laplace equation and boundary conditions (10a) are solved directly in discrete form in the fluid domain, and not—as was done in [2]—by constructing the velocity potential such that satisfies the governing equations identically in the domain. Casting the Laplace problem in physical space for clarity, this approach can be illustrated by considering a piston type wavemaker with position $x_1 = F(t)$, i.e. a translation of the wall. The inhomogeneous Neumann condition at the wavemaker (12) must be satisfied by the generation potential,

$$\partial_t F = \partial_{x_1} \phi_G, \quad x_1 = F, \quad (14)$$

while ϕ_G at the same time may not introduce inhomogeneous Neumann terms on the other walls of the domain. However, ϕ_G is not required to satisfy the Laplace equation, bottom or free surface boundary conditions, suggesting that ϕ_G can be expressed as,

$$\phi_G(x_i, t) = \partial_t F(x_2, x_3, t) \varphi_G(x_1, t), \quad (15)$$

where the function $\varphi_G(x_1, t)$ must satisfy $\partial_{x_1} \varphi_G|_{x_1=F(t)} = 1$ and $\partial_{x_1} \varphi_G|_{x_1=L_1} = 0$ in order to reflect the Neumann conditions at these locations. These conditions may be satisfied by the third order Hermite polynomial,

$$\varphi_G(x_1, t) = (L_1 - F(t))^{-2} (x_1 - F(t)) (L_1 - x_1)^2, \quad (16)$$

which further satisfies $\varphi_G|_{x_1=F(t)} = \varphi_G|_{x_1=L_1} = 0$. Inserting the splitting of the velocity potential (13) in (2) yields a Poisson equation for the propagation potential ϕ_P with homogeneous Neumann boundary conditions at the wavemaker and all fixed walls,

$$\phi_P = \phi - \phi_G, \quad x_3 = \zeta, \quad (17a)$$

$$\partial_{x_i x_i} \phi_P = -\partial_{x_i x_i} \phi_G, \quad x_i \in \Omega, \quad (17b)$$

$$\partial_{x_1} \phi_P = 0, \quad x_1 = F, \quad (17c)$$

$$n_i^{wall} \partial_{x_i} \phi_P = 0, \quad x_i \in \partial\Omega^{wall}, \quad (17d)$$

$$n_i^{bot} \partial_{x_i} \phi_P = -n_i^{bot} \partial_{x_i} \phi_G, \quad x_i \in \partial\Omega^{bot}, \quad (17e)$$

This Poisson equation may be discretized using the presented hybrid-spectral method and solved efficiently with the PDC method. The generation potential is thus essentially obtained free of cost, as only the action of the discrete Laplace operator on the generation potential must be evaluated in order to form the right-hand-side vector to the Poisson problem. The approach generalises to other types of wavemakers such as flaps, in which case it is necessary to express the generation potential and Poisson equation for the propagation potential in computational

space. Further the idea of additional potentials may be used to introduce Sommerfeld type, radiation outflow conditions at the opposite end of the wave tank.

Results & Future Work

The presented hybrid-spectral model provides an efficient and accurate tool for simulation of fully nonlinear free surface waves. Compared to the OceanWave3D finite difference based model, preliminary studies considering highly nonlinear propagating waves indicate both improved accuracy and a 50% reduction of the computational costs for identical problem sizes, see [4].

At the 28th IWWFEB we will present results obtained with the fully nonlinear numerical wave tank, including the wavemaker model by introduction of additional potentials essentially free of cost.

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