Semi analyticial soultion for second order hydroelastic response of the vertical circular cylinder in monochromatic water waves

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Theory

The body motion is described by the following displacement vector field:

$$\boldsymbol{H}(\boldsymbol{x},t) = \varepsilon \boldsymbol{H}^{(1)}(\boldsymbol{x},t) + \varepsilon^2 \boldsymbol{H}^{(2)}(\boldsymbol{x},t) = \sum_{i=1}^{N} \left\{ \varepsilon \xi_i^{(1)}(t) + \varepsilon^2 \xi_i^{(2)}(t) \right\} \boldsymbol{h}_i(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{h}_i(\boldsymbol{x})$ is the i-th mode shape function vector and ξ_i is its amplitude and N is the number of modes. The problem is formulated within the classical assumptions of the potential flow theory leading to the definition of the velocity potential $\Phi(\boldsymbol{x},t)$ for which the corresponding Boundary Value Problem (BVP) is built. Within the second order theory, which is of concern here, the fully non linear potential is formally written in the form:

$$\Phi(\boldsymbol{x},t) = \varepsilon \Phi^{(1)}(\boldsymbol{x},t) + \varepsilon^2 \Phi^{(2)}(\boldsymbol{x},t) \tag{2}$$

The use of the classical perturbation theory leads to the BVP's for different potentials. At first two orders, these BVP's are composed of the Laplace equation in the fluid domain, zero normal velocity at the fixed boundaries, radiation condition at infinity and the following free surface boundary conditions:

$$\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g \frac{\partial \Phi^{(1)}}{\partial z} = 0 \tag{3}$$

$$\frac{\partial^2 \Phi^{(2)}}{\partial t^2} + g \frac{\partial \Phi^{(2)}}{\partial z} = \frac{1}{g} \frac{\partial \Phi^{(1)}}{\partial t} [\frac{\partial^{(3)} \Phi^{(1)}}{\partial t^2 \partial z} + g \frac{\partial^2 \Phi^{(1)}}{\partial z^2}] - 2 \nabla \Phi^{(1)} \nabla \frac{\partial \Phi^{(1)}}{\partial t} \tag{4}$$

The corresponding body boundary condition is obtained after careful investigation of the body kinematics:

$$\nabla \Phi^{(1)} \boldsymbol{n} = \dot{\boldsymbol{H}}^{(1)} \boldsymbol{n} \tag{5}$$

$$\nabla \Phi^{(2)} \boldsymbol{n} = \dot{\boldsymbol{H}}^{(2)} \boldsymbol{n} - [(\boldsymbol{H}^{(1)} \nabla) \nabla \Phi^{(1)}] \boldsymbol{n} + (\dot{\boldsymbol{H}}^{(1)} - \nabla \Phi^{(1)}) \boldsymbol{n}^{(1)} \tag{6}$$

where overdot denotes the time derivative, $\boldsymbol{n}$ is the normal vector at rest and $\boldsymbol{n}^{(1)}$ its first order correction:

$$\boldsymbol{n}^{(1)} = (\nabla \boldsymbol{H}^{(1)}) \boldsymbol{n} - (\underline{\underline{\nabla \boldsymbol{H}^{(1)}}})^T \boldsymbol{n} \tag{7}$$

Finally, the total potential at each order is composed of the incident wave potential Φ_I and the perturbaion potential Φ_B which results from the interaction of the incident potential and the body:

$$\Phi(\boldsymbol{x},t) = \Phi_I(\boldsymbol{x},t) + \Phi_B(\boldsymbol{x},t) = \varepsilon[\Phi_I^{(1)}(\boldsymbol{x},t) + \Phi_B^{(1)}(\boldsymbol{x},t)] + \varepsilon^2[\Phi_I^{(2)}(\boldsymbol{x},t) + \Phi_B^{(2)}(\boldsymbol{x},t)] \tag{8}$$

The incident potential represents the excitation of the system and does not depends on the presence of the body. This means that it satisfies the Laplace equation in the fluid domain, zero flux condition at the bottom and the free surface conditions (3) and (4) where only the incident velocity potential is included. In order to solve for the amplitudes of the body deformations at each order, we need to further decompose the interaction potential Φ_B into the part Φ_R which depends directly on the body motion and the part Φ_D which is independent of the body motion at the corresponding order. In that respect, the radiation potentials are chosen to satisfy the homogeneous free surface boundary condition and the following body boundary conditions:

$$\nabla \Phi_R^{(1)} \boldsymbol{n} = \dot{\boldsymbol{H}}^{(1)} \boldsymbol{n} \qquad , \qquad \nabla \Phi_R^{(2)} \boldsymbol{n} = \dot{\boldsymbol{H}}^{(2)} \boldsymbol{n} \tag{9}$$

On the other side the potential Φ_D is chosen to satisfy all the remaining boundary conditions i.e. the non-homogeneous free surface condition and the remaining part of the body boundary condition. This leads to the following decomposition of the interaction potential Φ_B:

$$\Phi_B^{(1)}(\boldsymbol{x},t) = \phi_D^{(1)}(\boldsymbol{x},t) + \sum_{j=1}^{N} \dot{\xi}_j^{(1)} \Phi_{Rj}^{(1)} \qquad , \qquad \Phi_B^{(2)}(\boldsymbol{x},t) = \phi_D^{(2)}(\boldsymbol{x},t) + \sum_{j=1}^{N} \dot{\xi}_j^{(2)} \Phi_{Rj}^{(2)} \tag{10}$$

where Φ_D is usually called the diffraction potential and Φ_{Rj} the radiation potentials.
Let us also note that the diffraction potential is usually decomposed into two parts: ϕ_{DB} satisfying the homogeneous condition on the free surface and non-homogeneous condition on the body, and Φ_{DD} satisfying the non-homogeneous condition on the free surface and homogeneous on the body.
Once the different potentials evaluated, the pressure is calculated from Bernoulli equation:

$$p = -\varrho[gz + \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2] = -\varrho\{gz + \varepsilon\frac{\partial \Phi^{(1)}}{\partial t} + \varepsilon^2[\frac{\partial \Phi^{(2)}}{\partial t} + \frac{1}{2}(\nabla\Phi^{(1)})^2 + (\boldsymbol{H}^{(1)}\nabla)\frac{\partial \Phi^{(1)}}{\partial t}]\} \tag{11}$$

Frequency domain

The above defined problem is now formulated in frequency domain. We start by defining the incident wave potential:

$$\Phi_I(\boldsymbol{x},t) = \Re\{\varphi_I^{(1)}(\boldsymbol{x})e^{-i\omega t}\} + \Re\{\varphi_I^{(2)}(\boldsymbol{x})e^{-2i\omega t}\} \tag{12}$$

where:

$$\varphi_I^{(1)} = -\frac{igA}{\omega}\frac{\cosh k_0(z+D)}{\cosh k_0 D}e^{ik_0x} \qquad , \qquad \varphi_I^{(2)} = -\frac{3i\omega\nu A^2}{2\sinh^2 k_0 D}\frac{\cosh 2k_0(z+D)}{4\nu\sinh^2 k_0 D}e^{ik_0x} \tag{13}$$

The boundary conditions for other potentials follow straightforwardly and we end up with the fact taht all the potentals, at any order, satisfy one of the two types of the BVP defined below:

$$\left.\begin{array}{lll}
\Delta\psi_B = 0 & \Delta\psi_Q = 0 & r > a\ ,\ -D < z < 0 \\
-\alpha\psi_B + \dfrac{\partial \psi_B}{\partial z} = 0 & -\alpha\psi_Q + \dfrac{\partial \psi_Q}{\partial z} = Q(r,\theta) & z = 0 \\
\dfrac{\partial \psi_B}{\partial n} = v(z,\theta) & \dfrac{\partial \psi_Q}{\partial n} = 0 & r = a \\
\dfrac{\partial \psi_B}{\partial z} = 0 & \dfrac{\partial \psi_Q}{\partial z} = 0 & z = -D \\
\psi_B \to 0 & \psi_Q \to 0 & r \to \infty
\end{array}\right\} \tag{14}$$

The potential ψ_B is called body perturbation potential and ψ_Q free surface perturbation potential and they can be calculated either numerically or semi-analytically for vertical circular cylinder (e.g. see [1]).

Hydrodynamic forces and body motions

The forces are obtained by integration of the pressure over the wetted body surface:

$$\boldsymbol{F} = \int\int_{\tilde{S}_b} p\boldsymbol{H}\tilde{\boldsymbol{n}}dS \tag{15}$$

where p is the pressure calculated from Bernoulli equation (11), $\tilde{S}_b$ is the instantaneous body surface and $\tilde{\boldsymbol{n}}$ is the instantaneous normal vector. Special attention should be given to the proper separation of different terms in order to write the final motion equation:

$$\{-\omega^2([\boldsymbol{M}] + [\boldsymbol{A}(\omega)]) - i\omega[\boldsymbol{B}(\omega)] + [\mathbf{C}]\}\{\xi^{(1)}\} = \{\boldsymbol{F}_E^{(1)}\} \tag{16}$$

$$\{-4\omega^2([\boldsymbol{M}] + [\boldsymbol{A}(2\omega)]) - 2i\omega[\boldsymbol{B}(2\omega)] + [\mathbf{C}]\}\{\xi^{(2)}\} = \{\boldsymbol{F}_E^{(2)}\} \tag{17}$$

where $[\boldsymbol{M}]$ is the modal mass matrix, $[\boldsymbol{A}]$ is the associated added mass matrix, $[\boldsymbol{B}]$ is the damping matrix, $[\mathbf{C}]$ is the stiffnes matrix (including both hydrostatic and structural parts) and $\{\boldsymbol{F}_E^{(1)}\}$ and $\{\boldsymbol{F}_E^{(2)}\}$ are the first and second order excitation forces. Note that the added mass and damping matrices are obtained by integrating the pressure associated with the radiation potential φ_{Rj} while the excitation forces are obtained after integration of all the remaining pressure components.

Vertical circular cylinder

General solution for ψ_B and ψ_Q can be written in the following form [1]:

$$\psi_B(r,z,\theta) \quad = \quad \sum_{m=-\infty}^{\infty}\left[f_0(z)\beta_{m0}H_m(k_0r)+\sum_{n=1}^{\infty}f_n(z)\beta_{mn}K_m(k_nr)\right]e^{im\theta} \tag{18}$$

$$\psi_Q(r,z,\theta) \quad = \quad \sum_{m=-\infty}^{\infty}\left[f_0(z)A_{m0}+\sum_{n=1}^{\infty}f_n(z)A_{mn}\right]e^{im\theta} \tag{19}$$

where the most complex terms which invloves the infinite integration over the free surface are given by:

$$A_{m0}=-\frac{2C_0\int_a^\infty H_m(k_0\rho)Q_m(\rho)\rho d\rho}{k_0aH'_m(k_0a)} \quad , \quad A_{mn}=-\frac{2C_0\int_a^\infty K_m(k_n\rho)Q_m(\rho)\rho d\rho}{k_naK'_m(k_na)} \tag{20}$$

and the detailed expressions for all the other terms can be found in [1].
In order to solve for ψ_B and ψ_Q we need to express the boundary conditions in cylindrical coordinates. First we assume that the column is free to bend only and we define the deformation modes:

$$\boldsymbol{h}_i=h_{ix}(z)\boldsymbol{i}+0\boldsymbol{j}+0\boldsymbol{k} \quad , \quad \boldsymbol{h}_i\nabla=h_{ix}(z)\frac{\partial}{\partial x}=h_{ix}(z)(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}) \tag{21}$$

We also assume that the normal vector is pointing out of the fluid domain and we write:

$$\boldsymbol{n}=-\cos\theta\boldsymbol{i}-\sin\theta\boldsymbol{j}+0\boldsymbol{k}=-\boldsymbol{e}_r+0\boldsymbol{e}_\theta+0\boldsymbol{k} \quad , \quad \boldsymbol{n}^{(1)}=\sum_{i=1}^{N}\xi_i^{(1)}\frac{\partial h_{ix}}{\partial z}\cos\theta\boldsymbol{k} \tag{22}$$

where $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{k})$ are the unit vectors of the cylindrical coordinate system.

Body boundary conditions

After applying the described theory we end up with the following body boundary conditions (only non-zero terms are presented):

$$\frac{\partial\varphi_{DB}^{(1)}}{\partial n} \quad = \quad -\frac{\partial\varphi_I^{(1)}}{\partial n} \quad , \quad \frac{\partial\varphi_{Rj}^{(1)}}{\partial n}=\boldsymbol{h}_j\boldsymbol{n}=-h_{jx}\cos\theta \tag{23}$$

$$\frac{\partial\varphi_{DB}^{(2)}}{\partial n} \quad = \quad -\frac{\partial\varphi_I^{(2)}}{\partial n} \tag{24}$$

$$+\frac{1}{2}\sum_{i=1}^{N}\xi_i^{(1)}h_{ix}(z)(\frac{\partial^2\varphi^{(1)}}{\partial r^2}\cos\theta-\frac{1}{r}\frac{\partial^2\varphi^{(1)}}{\partial\theta\partial r}\sin\theta)-\frac{1}{2}\sum_{i=1}^{N}\xi_i^{(1)}\frac{\partial h_{ix}}{\partial z}\frac{\partial\varphi^{(1)}}{\partial z}\cos\theta \tag{25}$$

$$\frac{\partial\varphi_{Rj}^{(2)}}{\partial n} \quad = \quad \boldsymbol{h}_j\boldsymbol{n}=-h_{jx}\cos\theta \tag{26}$$

Free surace boundary conditions

All the free surface conditions are homogeneous except the one for $\varphi_{DD}^{(2)}$. The non-homogeneous term $Q_{DD}^{(2)}$ can be written as follows:

$$Q_{DD}^{(2)}=\frac{i\omega}{g}\{(\nabla\varphi_B^{(1)}\nabla\varphi_B^{(1)}+2\nabla\varphi_I^{(1)}\nabla\varphi_B^{(1)})-\frac{1}{2}[(\varphi_B^{(1)}+\varphi_I^{(1)})(\frac{\partial^2\varphi_B^{(1)}}{\partial z^2}-\nu^2\varphi_B^{(1)})+\varphi_B^{(1)}(\frac{\partial^2\varphi_I^{(1)}}{\partial z^2}-\nu^2\varphi_I^{(1)})]\} \tag{27}$$

In the case of the vertical circular cylinder, the first order interaction potential $\varphi_B^{(1)}$ can be written in the following form:

$$\varphi_B^{(1)}=\sum_{m=-\infty}^{\infty}\left[f_0(z)\gamma_{m0}H_m(k_0r)+\sum_{n=1}^{\infty}f_n(z)\gamma_{mn}K_m(k_nr)\right]e^{im\theta}=\sum_{m=-\infty}^{\infty}f_0(z)\gamma_{m0}H_m(k_0R)e^{im\theta} \tag{28}$$

where the second expression is valid at large radial distance R only.
With this in mind, we can calculate $Q_{DDm}^{(2)}$ and subsequently the second order diffraction potential $\varphi_{DD}^{(2)}$. Special attention should be given to the evaluation of the free surface integrals in (20) which are highly oscillatory and extend to infinity. In this work we use the numerical integration close to the cylinder and the semi-analytical method [1] in the far field.

Numerical example

We chose the example proposed in [2]. The following modes of deformation are defined:

$$h_{ix}(z) = q^2 P_{i-1}(q) \quad , \quad q = 1 + z/D \quad , \quad P_n(q) = \sum_{m=0}^{n} (-1)^m \frac{(4+2n-m)!}{m!(n-m)!(4+n-m)!} q^{n-m} \tag{29}$$

The deformation modes and the linear RAO results are shown in Figure 1. These results are the same as the numerical results given in [2]. This validates the present approach for linear case.
The preliminary second order results are shown in Figure 2. They concern the non-homogeneous term in the second order free surface condition and the difference in between the results, when full and asymptotic expressions for first order potential are used, is shown. We can see that the results converge quickly to asymptotic solution which is good point which means that the classical procedure for second order diffraction [1] can be used very quickly. Knowing that the pure second order diffraction problem for vertical circular cylinder is already solved [1], this ensures the efficiency of the proposed method. More detailed results will be presented at the Workshop.

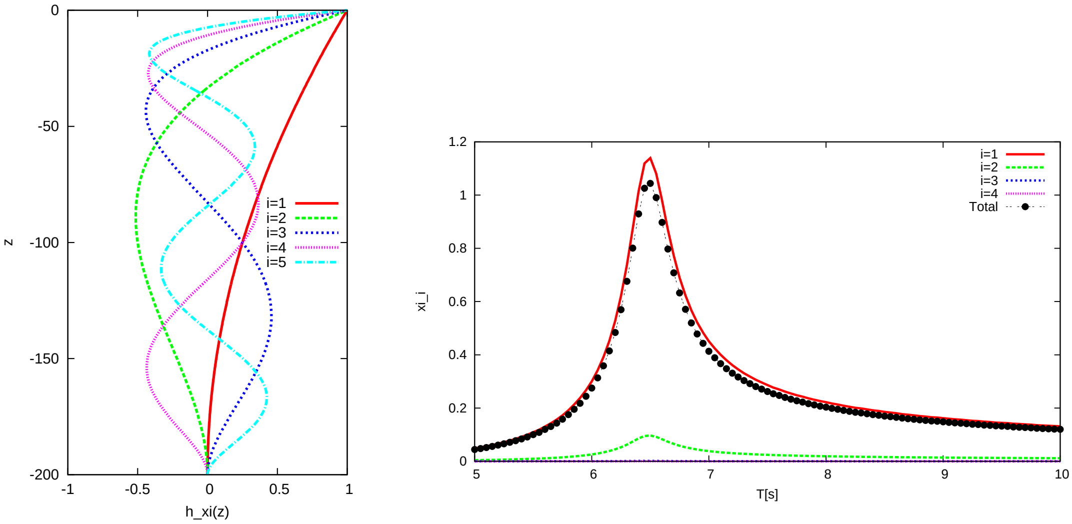

Figure 1: Deformation modes and linear RAO of the motion of the column top.

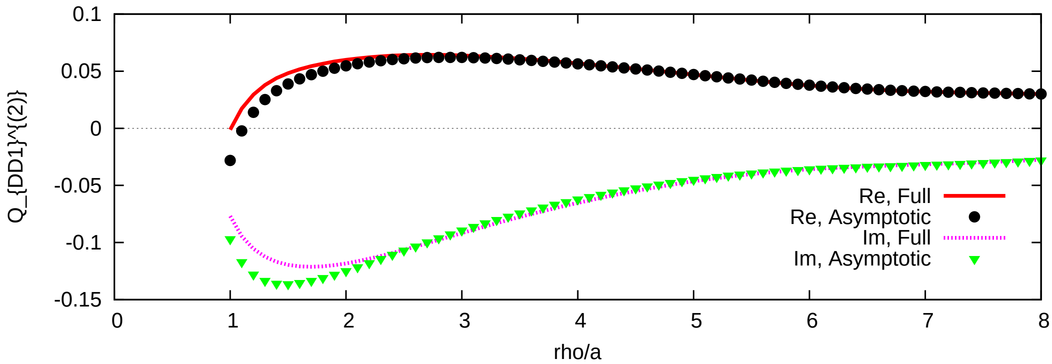

Figure 2: Original and asymptotic second order forcing term on the free surface.

Acknowledgements

The authors acknowledge the support of the National Research Foundation of Korea (NRF) grant funded by the Korea Government (MEST) through GCRC-SOP.

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