On the wave resistance of an immersed prolate spheroid in infinite water depth

by

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1 Introduction

It is the purpose of this study to present a newly developed robust and efficient solution of the wave resistance problem of immersed prolate spheroids moving under regular waves with constant forward speed. Here the term "wave resistance" is associated with the coupled forward speed and wave impact problem, the latter being directed to the spheroid under arbitrary heading angle. Parts of the investigated subject were partly treated in the past by several authors who, however, isolated the two major contributions, namely the forward speed and the wave effects. In this context Havelock [1] approximated the wave resistance of prolate and oblate spheroids using Lagally's [2] theorem (without mentioning Lagally in this connection) using the axial source distribution corresponding to the motion of the spheroid in an infinite mass of liquid. Farell [3] expanded the sources distributed on the surface of the spheroid into series of spheroidal harmonics and reported significant differentiations compared to Havelock's [1] predictions. Wu and Eatock Taylor [4-5] used Farell's [3] approach to tackle the diffraction (only) problem assuming (only) frontal wave heading.

Here a solution to the complete problem is presented (waves and forward speed) which we achieved by employing Miloh's [6] image singularities.

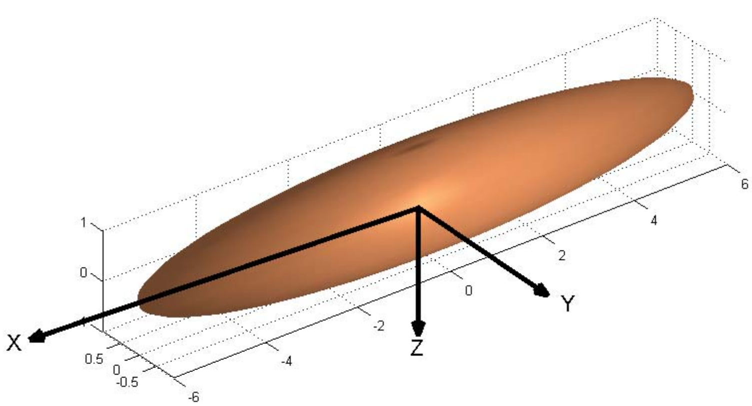

Fig. 1 3D image of a prolate spheroid below the free surface (non-axisymmetric case) with a/b=6.

2 Multipole expansions in curvilinear coordinates for surface waves

The concerned spheroidal body is considered immersed at a distance f below the undisturbed free surface (Fig. 1). The non-axisymmetric case is considered meaning that symmetrical axis (x-axis) is parallel to the free surface. Using left-handed Cartesian (x, y, z) coordinates, fixed on the free surface with z pointing downwards we start with the following well known Fourier expansion for the fundamental Green's function of the Laplace equation

$$\frac{1}{\sqrt{x^2+y^2+(z-f)^2}}=\frac{1}{2\pi}\int_0^{\infty}\int_{-\pi}^{\pi} e^{-k|z-f|+ik(x\cos a+y\sin a)}\mathrm{d}a\mathrm{d}k \qquad (1)$$

The common linearized free-surface boundary condition for the velocity potential, in the case of time-harmonic oscillations with frequency ω, including a forward motion with constant velocity U along the x direction, is

$$\left(-i\omega+U\frac{\partial}{\partial x}\right)^2\phi-g\frac{\partial\phi}{\partial z}=0, \tag{2}$$

to be applied at z=0 with g being the gravitational acceleration. Eq. (2) can also be cast to

$$\left(\sqrt{K}-k\sqrt{\tau}\cos a\right)^2\phi+\frac{\partial\phi}{\partial z}=0, \tag{3}$$

where $\tau=U^2/g$ and $K=\omega^2/g$. Hence the Green's function can be written as

$$G(x,y,z)=\frac{1}{\sqrt{x^2+y^2+(z-f)^2}}-\frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}Q(k,a)e^{-k(z+f)+ik(x\cos a+y\sin a)}\mathrm{d}a\mathrm{d}k, \tag{4}$$

where

$$Q(k,a)=\frac{\left(\sqrt{K}-k\sqrt{\tau}\cos a\right)^2+k}{\left(\sqrt{K}-k\sqrt{\tau}\cos a\right)^2-k} \tag{5}$$

Clearly for τ=0 (U=0) we recover the diffraction problem whereas for K=0 (ω=0) we get the forward speed (wave resistance) problem. The former case applies directly to the radiation problem as well by replacing the Neumann boundary condition on the spheroid by

$$\frac{\partial\phi_R^{(i)}}{\partial n}=n_i\,;\ \frac{\partial\phi_R^{(i+3)}}{\partial n}=\left(\vec{r}\times\vec{n}\right)_i,\ \text{for } i=1,2,3, \tag{6}$$

where the index R is used to denote the radiation potential, $\vec{r}$ is the position vector (with respect to the origin) and n_i denotes the unit vector in the direction i. The boundary conditions involved in (6) describe all six radiation problems. Finally, in order to satisfy the far-field Sommerfeld radiation condition, the following notation is used in (4) and in the sequel,

$$\int_0^\infty\frac{F(k)}{\sigma-k}\mathrm{d}k=PV\int_0^\infty\frac{F(k)}{\sigma-k}\mathrm{d}k-i\pi F(\sigma), \tag{7}$$

where the acronym PV is used to denote Cauchy's Principal Value Integral.

3 Prolate spheroidal coordinates – non axisymmetric case

A prolate spheroidal (ζ, μ, ψ) coordinate system is defined, with an origin at a depth f below the undisturbed free surface such that the spheroid is assumed to be fully submerged. We also define $\zeta=\cosh u$, $\mu=\cos\vartheta$, where $0\le u\le\infty$, $0\le\vartheta\le\pi$ and $0\le\psi\le2\pi$. The transformation from prolate spheroidal to Cartesian coordinates is $x=c\cosh u\cos\vartheta$, $y=c\sinh u\sin\vartheta\sin\psi$, $z=c\sinh u\sin\vartheta\cos\psi$. Hence, $x=c\mu\zeta$ and $z+iy=c(\zeta^2-1)^{1/2}(1-\mu^2)^{1/2}e^{i\psi}$, where c represents half the distance between the two foci of the spheroid. In terms of the semi-major a and semi-minor b axes of the spheroid, c is expressed as $c=ae$ where $e=(1-(b/a)^2)^{1/2}$ denotes the eccentricity. It is noted that in the following analysis c was taken equal to unity and thus it is used as a reference length scale. For manipulating the original Green's function (see Eq. (4)) in prolate spheroidal coordinates we use a most useful relation which was originally suggested without proof by Havelock [7] and later rigorously obtained by Miloh [6]. The concerned relation expresses any exterior spheroidal harmonic in terms of prescribed singularities disturbed on the major axis of the spheroid between the two foci. In particular Havelock's formula for the present case may be written as

$$P_n^m(\mu)Q_n^m(\zeta)e^{im\psi}=\frac{1}{2}\left(\frac{\partial}{\partial z}+i\frac{\partial}{\partial y}\right)^m\int_{-1}^{1}\frac{(1-\lambda^2)^{m/2}P_n^m(\lambda)}{\sqrt{(x-\lambda)^2+y^2+z^2}}\mathrm{d}\lambda \tag{8}$$

where P_n^m, Q_n^m denote the associate Legendre functions of the first and the second kind with order m and degree n. Accordingly, using the proposed approach one can readily express the Green's function as

$$G_n^m(x,y,z) = P_n^m(\mu)Q_n^m(\zeta)e^{im\psi} - \frac{(-1)^m}{4\pi}\int_0^\infty \mathrm{f}_m^n(x,y;k)k^m Q(k,a)e^{-k(f-z)}\mathrm{d}k \tag{9}$$

where

$$\mathrm{f}_m^n(x,y;k) = \int_{-\pi}^{\pi}\int_{-1}^{1}(1-\lambda^2)^{m/2}P_n^m(\lambda)e^{-ik\lambda\cos a}(\sin a-1)^m e^{ik(x\cos a+y\sin a)}\mathrm{d}\lambda\mathrm{d}a \tag{10}$$

It should be noted that the above relations assume that the direction of the vertical coordinate has been reversed showing upward. To enable feasibility of numerical computations Eqs. (9)-(10) should be further manipulated. To this end exp($-ik\lambda\cos a$) is expanded into a Taylor series and a most useful relation that can be found in Gradshteyn and Ryzhik [8, p. 772] is employed. Hence the next representation of Green's function reads

$$\begin{aligned} G_n^m(x,y,z) = P_n^m(\mu)Q_n^m(\zeta)e^{im\psi} - \frac{(-1)^m}{4\pi}\sum_{q=0}^{\infty}(-i)^q I_n^m(q)\int_0^\infty k^{m+q}e^{-2kf} \\ \times\int_{-\pi}^{\pi}Q(k,a)(\cos a)^q(\sin a-1)^m e^{kz^*+ik(x\cos a+y\sin a)}\mathrm{d}a\mathrm{d}k \end{aligned} \tag{11}$$

where

$$I_n^m(q) = \frac{(1-(-1)^{n+q})(-1)^m 2^{-m-1}\Gamma(1/2+q/2)\Gamma(1+q/2)\Gamma(1+m+n)}{q!\Gamma(1-m+n)\Gamma(1+m/2-n/2+q/2)\Gamma(3/2+m/2+n/2+q/2)} \tag{12}$$

Note that the dependence on the vertical coordinate was transformed to be expressed in terms of the z^* axis fixed on the center of the body.

In order to allow the employment of the zero velocity condition on body's surface the exponential term in Eq. (11) must be cast to spheroidal harmonics. To this end a most useful relation shown by Havelock [9] will be applied. This is

$$e^{kz^*+ik(x\cos a+y\sin a)} = \sum_{s=0}^{\infty}\sum_{t=0}^{s}\frac{\varepsilon_t}{2}i^{s-t}(2s+1)\frac{(s-t)!}{(s+t)!}j_s(kc\cos a)P_s^t(\mu)P_s^t(\zeta)\left[N_t(a)\cos t\psi + i\tilde{N}_t(a)\sin t\psi\right] \tag{13}$$

where j_s is the spherical Bessel function of the first kind and

$$N_t(a) = \left[\left(\frac{1+\sin a}{\cos a}\right)^t + \left(\frac{1-\sin a}{\cos a}\right)^t\right],\ \tilde{N}_t(a) = \left[\left(\frac{1-\sin a}{\cos a}\right)^t - \left(\frac{1+\sin a}{\cos a}\right)^t\right] \tag{14}$$

The final form of Green's function now becomes

$$G_n^m(x,y,z) = P_n^m(\mu)Q_n^m(\zeta)e^{im\psi} + \sum_{s=0}^{\infty}\sum_{t=0}^{s}\left(C_{ns}^{mt}\cos t\psi + i\tilde{C}_{ns}^{mt}\sin t\psi\right)P_s^t(\mu)P_s^t(\zeta) \tag{15}$$

where

$$\begin{aligned} C_{ns}^{mt} = -\frac{(-1)^m}{4\pi}\frac{\varepsilon_t}{2}i^{s-t}(2s+1)\frac{(s-t)!}{(s+t)!}\sum_{q=0}^{\infty}(-i)^q I_n^m(q)\int_{-\pi}^{\pi}(\cos a)^q(\sin a-1)^m N_t(a) \\ \times\int_0^\infty k^{m+q}e^{-2kf}Q(k,a)j_s(kc\cos a)\mathrm{d}k\mathrm{d}a \end{aligned} \tag{16}$$

and $\tilde{C}_{ns}^{mt}$ is obtained through the former after replacing $N_t(a)$ by $\tilde{N}_t(a)$. The calculation of C_{ns}^{mt} and $\tilde{C}_{ns}^{mt}$ coefficients virtually completes the solution to the problem and the scattered potential is obtained by the following multipole expansion in prolate spheroidal coordinates

$$\phi(\mu,\zeta,\psi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}A_n^m\left\{P_n^m(\mu)Q_n^m(\zeta)e^{im\psi} + \sum_{s=0}^{\infty}\sum_{t=0}^{s}\left(C_{ns}^{mt}\cos t\psi + i\tilde{C}_{ns}^{mt}\sin t\psi\right)P_s^t(\mu)P_s^t(\zeta)\right\} \tag{17}$$

The unknown expansion coefficients A_n^m are derived by employing the zero velocity condition on the wetted surface of the spheroid. Eq. (17) represents a global formulation that covers all possible cases, namely the diffraction problem (U=0), the wave resistance problem (ω=0) and the complete forward speed and wave impact problem ($U\neq 0$, $\omega\neq 0$). In fact only $Q(k,a)$ need to be changed whilst for the diffraction and the wave resistance problems is reduced respectively to

$$Q(k,a)=\frac{K+k}{K-k},\ Q(k,a)=\frac{k+1/\tau\cos^2 a}{k-1/\tau\cos^2 a} \tag{18}$$

The most challenging part as regards the computation of C_{ns}^{mt} and $\widetilde{C}_{ns}^{mt}$ is the numerical evaluation of Cauchy's Principal Value Integral involved in the infinite series of Eq. (16). However special attention must be given to the singularity that occurs at $a=\pm\pi/2$. For the diffraction problem this can be easily avoided adopting the ascending series of Bessel function. This however will make the expressions that provide C_{ns}^{mt} and $\widetilde{C}_{ns}^{mt}$ much more complicated. In particular the former will be given by

$$\begin{aligned}C_{ns}^{mt} &= \frac{(-1)^m i^{s-t}}{4\pi}\pi^{1/2}\frac{\varepsilon_t}{2}(2s+1)\frac{(s-t)!}{(s+t)!}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{p+q} i^q (1/2)^{2p+s+1}}{q!\,p!\,\Gamma(s+p+3/2)} I_n^m(q)\\ &\times\int_0^\infty k^{m+q+2p+s}\frac{K+k}{K-k}e^{-2kf}\,\mathrm{d}k\int_{-\pi}^{\pi}(\sin a-1)^m(\cos a)^{q+2p+s-t}\left[(1+\sin a)^t+(1-\sin a)^t\right]\mathrm{d}a\end{aligned} \tag{19}$$

that allows separate integrations oven a and k. For the coupled forward speed and wave heading problem one must find the roots of the denominator in Eq. (5). In that case the numerical implementation requires the calculation of two Cauchy PV integrals in terms of the roots

$$\rho_{1,2}=\frac{2\sqrt{K\tau}\cos a+1\pm\left(1+4\sqrt{K\tau}\cos a\right)^{1/2}}{2\tau\cos^2 a} \tag{20}$$

The roots $\rho_{1,2}$ must be real regardless a. This condition is always satisfied if $\omega U/g<1/4$ and that finding is in compliance with the requirement for the existence of an upper bound for the critical frequency $\omega_c=0.25g/U$ of oscillating singularities [10] where the classical linearized solution yields infinitely large wave amplitude.

Some numerical examples for prolate spheroids of different submergence depth and slenderness ratio will be presented at the Workshop together with a comparison against Farell's [3] and Havelock's [1] approximations.

4 References

[1] Havelock TH (1931) The wave resistance of a spheroid. Proc Royal Soc London, A131: 275-285

[2] Lagally M (1922) Berechnung der Kräfte und Momente, die strömende Fliassigkeiten auf ihre Begrenzung Ausiiben. Z Angew Math Mech, 2: 409-422

[3] Farell C (1973) On the wave resistance of a submerged spheroid. J Ship Res, 17: 1-11

[4] Wu GX, Eatock Taylor R (1987) The exciting force on a submerged spheroid in regular waves. J Fluid Mech, 182: 411-426

[5] Wu GX, Eatock Taylor R (1989) On radiation and diffraction of surface waves by submerged spheroids. J Ship Res, 33: 84-92

[6] Miloh T (1974) The ultimate image singularities for external ellipsoidal harmonics. SIAM J Appl Math 26: 334-344

[7] Havelock TH (1952) The moment on a submerged solid of revolution moving horizontally. Quart. J. Mech Appl Math, 5: 129-136

[8] Gradshteyn IS, Ryzhik IM (2007) Tables of integrals, series and products (seventh edition). Elsevier Academic Press, London

[9] Havelock TH (1954) The forces on a submerged body moving under waves. Transactions Institution of Naval Architects 96: 77-88

[10] Dagan G, Miloh T (1982) Free surface flow past oscillating singularities at resonance frequency. J Fluid Mech, 120, 139-154.