A comparaison of simulation approaches based on the Zakharov equations for nonlinear waves in the coastal zone

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Introduction

To model the propagation of nearshore waves, scientists and engineers have a need for accurate, rapid wave models that are capable of simulating wave fields over large spatial domains. Accurate models must also be capable of taking into account the nonlinear and dispersive effects of propagating waves, which are particularly important in the nearshore region.

A variety of models have been developed to address this need, ranging from models based on the mild slope equation for linear monochromatic waves to CFD approaches based on the Navier-Stokes equations for incompressible and turbulent flows, including models based on Boussinesq, Korteweg-DeVries, Serre, Green-Naghdi equations among others. The application of CFD models for modeling the nearshore zone is still largely limited by the size of the spatial domain due to the computational cost. Although these models are well-adapted to simulating wave-structure interactions at a local scale and wave breaking processes, alternative approaches are needed to model nearshore waves on larger spatial scales.

Potential wave models, based on the assumption of inviscid and irrotational flow, require the resolution of the Laplace equation in the fluid domain, with the correct specification of the boundary conditions. Fully nonlinear potential flow models have been developed following this approach, using boundary integral methods (e.g. [Grilli and Horrillo, 1999]) or finite differences (e.g. [Bingham and Zhang, 2007]) to obtain highly accurate results. The problem may be further simplified by assuming a form of the vertical structure of the flow, as in Boussinesq or Serre models, in which the computational domain has thus been reduced by one dimension. Boussinesq-type models were originally based on the assumptions of weak nonlinearity (ratio of wave height to wave length) and weak dispersive effects, limiting their range of application. However, significant developments in the last 25 years have allowed the extension of Boussinesq-type models to a include the effects of nonlinearity and dispersion to varying degrees.

Here, three different approaches for resolving the so-called Zakharov equations [Zakharov, 1968] in the case of variable bathymetry are compared for several test cases, two of them being presented hereafter: (i) the propagation and reflection from a wall of a solitary wave [Cooker et al., 1997] and (ii) the shoaling of a series of waves on a beach [Kennedy et al., 2001].

Overview of mathematical and numerical models

A three-dimensional domain $(\underline{x}, z)$ is considered, with a free surface elevation $z = \eta(\underline{x}, t)$ and a bottom boundary $z = -h(\underline{x})$, which are single-valued functions of $\underline{x}$. Assuming irrotational flow, the velocity potential $\phi(\underline{x}, z, t)$ satisfies the Laplace equation in the fluid domain:

$$\nabla^2\phi = 0, \qquad -h(\underline{x}) \leq z \leq \eta(\underline{x}, t) \tag{1}$$

Following Zakharov [1968], the kinematic and dynamic surface nonlinear boundary conditions are formulated as:

$$\eta_t = -\nabla\eta \cdot \nabla\tilde{\phi} + \tilde{w}(1 + \nabla\eta \cdot \nabla\eta) \tag{2}$$

$$\tilde{\phi}_t = -g\eta - \frac{1}{2}\nabla\tilde{\phi} \cdot \nabla\tilde{\phi} + \frac{1}{2}\tilde{w}^2(1 + \nabla\eta \cdot \nabla\eta), \tag{3}$$

where $\tilde{\phi}(\underset{\sim}{x}, t) = \phi(\underset{\sim}{x}, \eta(\underset{\sim}{x}, t), t)$, and $\tilde{w}(\underset{\sim}{x}, t) = \frac{\partial \phi}{\partial z}|_{z=\eta}$ is the vertical velocity at the surface. By specifying lateral boundary conditions and choosing a method to calculate the gradients of ϕ and η, these two equations can be integrated in time once the vertical velocity at the surface $\tilde{w}(\underset{\sim}{x}, t)$ has been determined. The problem requiring attention is the method used to calculate $\tilde{w}(\underset{\sim}{x}, t)$, typically called Dirichlet-to-Neumann (DtN), and three different approaches are described here.

1. **Model A.** The first method requires discretizing the vertical (with NZ points to form $NL = NZ - 1$ layers) to calculate ϕ in the entire domain using finite differences, following the approach of Bingham and Zhang [2007] and Engsig-Karup et al. [2009]. The vertical distribution of the points is defined by the zeros of Chebyshev-Gauss-Lobatto polynomials.
2. **Model B.** The second method is based on the spectral approach of Tian and Sato [2008], which expresses the vertical profile of ϕ as a linear combination of base functions (with NL being the maximum order of the polynomial), here defined to be the Chebyshev orthogonal polynomials of the first kind.
3. **Two-layer model.** The third approach is a two-layer Boussinesq-type model, developed by Chazel et al. [2009], with the vertical domain divided into two layers at the level $z = -\sigma h(\underset{\sim}{x})$, with σ defined to be a constant. A DtN operator is defined at the still water level, $z = 0$, and two closure relations are determined via truncated Taylor expansions of the velocity potential and vertical velocity between the still water level and the free surface. The generalized Boussinesq procedure is followed by searching for a solution to the Laplace equation in each layer in the form of a truncated Taylor series expansion in the vertical, in which Padé approximants are used to lower the order of derivatives in the truncated series.

All spatial derivatives are calculated using fourth-order finite difference schemes, and all simulations are completed using a fourth-order Runge-Kutta integration scheme.

Results on two selected test cases

Propagation and reflection from a wall of a solitary wave

The first test case is based on the work of Cooker et al. [1997] and Grilli and Svendsen [1991], who simulated the run-up and reflection from a wall of solitary waves with relative wave heights of H/h (H=wave height) ranging from 0.075 to 0.7. The initial conditions of the simulations are given by the iterative method of Tanaka [1986] for a water depth of $h = 1$ m. The non-dimensional maximum run-up η_{max}/h and the time of maximum run-up $t_{\eta_{max}}/\tau$ (with $\tau = (h/g)^{1/2}$) for each run are compared with the results of Cooker et al. [1997] (Figure 1).

The values of maximum run-up simulated by the three models agree well with each other and with the reference data until a relative wave height of 0.4 (Figure 1a). With more highly nonlinear waves, the simulations with the two-layer model begin to diverge from the reference data, underestimating the maximum level of run-up. Models A and B continue to show similar results until the relative wave height reaches 0.65 and 0.6, respectively. Above these limits, the simulations diverge before the waves reach η_{max}. The simulated times of maximum run-up (Figure 1b) agree well with the reference data except for the most extreme values of relative wave height, indicating that the models are able to simulate well the phase speed of the waves, except for the most nonlinear cases at the limit at which the simulations diverge.

Shoaling of a series of waves on a beach

The second test case is based on the work of Kennedy et al. [2001], who modeled the shoaling of a series of waves up a beach until the breaking point. The initial condition at the free surface

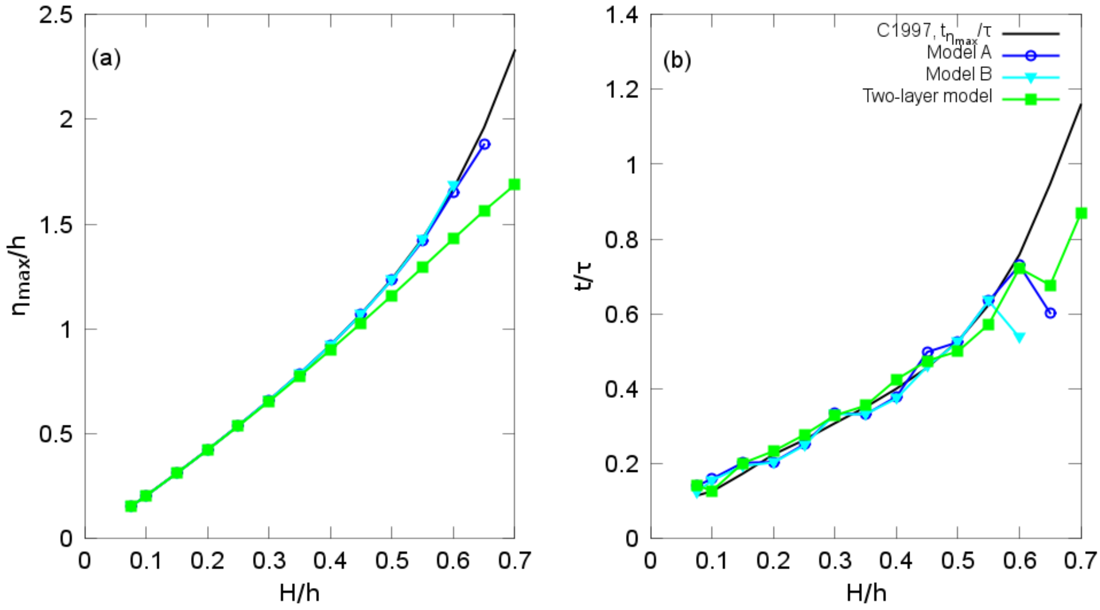

Figure 1: Comparison of (a) the maximum elevation of wave run-up and (b) the time that the wave reached the maximum run-up with the results of Cooker et al. [1997].

is defined by 10 sine waves of decreasing amplitude in the first half of the domain, with no initial velocity. When the simulations start, the waves begin propagating up the beach, and the simulations are stopped at a pre-defined time before wave breaking begins. Of the three cases evaluated by Kennedy et al. [2001], the results of the simulations with the most nonlinear waves, with a relative wave amplitude a/h=0.125 (a=wave amplitude), are presented here (Figure 2). The most difficult region to accurately simulate is the region of wave shoaling, from $x = 40$ m to $x = 60$ m, and the elevation of the maximum wave crests and minimum wave troughs is compared with the reference data of Kennedy et al. [2001].

For all three models, the simulation results depend highly on the horizontal resolution of the domain (Figure 2). Models A and B have nearly identical results, and an increase in the horizontal resolution from a constant $\Delta x = 0.1$ m to a variable $\Delta x = 0.025 - 0.1$ m enables both models to significantly improve the agreement with the reference data set. The two-layer Boussinesq model significantly underpredicts the maximum wave crest elevation with low horizontal resolution ($\Delta x = 0.2$ m). However, when the horizontal resolution is increased, instabilities develop and rapidly grow, causing the divergence of the model. It is necessary to apply a filter [Savitzky and Golay, 1964] for $\Delta x < 0.2$ m. Decreasing the resolution to $\Delta x = 0.11$ m does not improve the results as the application of the filter reduces the maximum crest heights.

These two test cases demonstrate the differences between the three strategies for resolving the Zakharov equations. Current work includes comparing the model simulations to laboratory observations of the propagation and shoaling of waves over a bar, and the result of additional cases will be presented during the Workshop.

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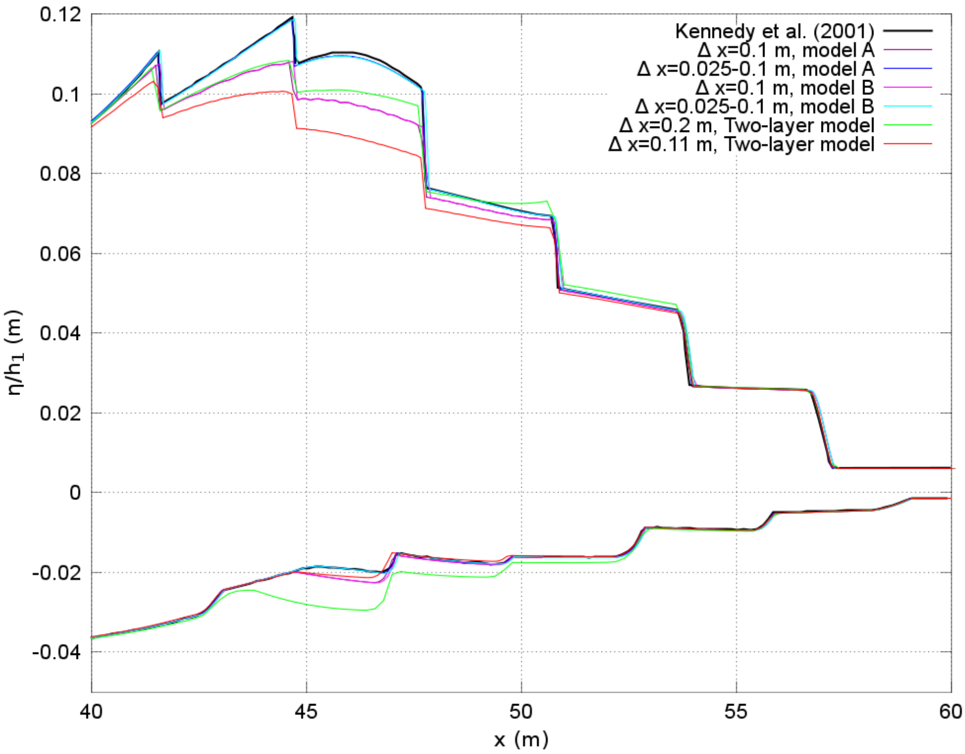

Figure 2: Comparison of the envelope of wave crests and wave troughs for the test case with a/h=0.125 with the results of Kennedy et al. [2001].

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