

**Numerical time integration methods  
for a point absorber wave energy converter**

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## 1 Background

The objective of this abstract is to provide a review of models for motion simulation of marine structures with a special emphasis on wave energy converters. The time-domain model is applied to a point absorber system working in pitch mode only. The device is similar to the well-known Wavestar float located in the Danish North Sea. The main objective is to produce a tool that can accurately simulate the dynamics of a floating structure with an arbitrary geometry provided the frequency domain coefficients are calculated beforehand. The latter calculation is based on linear fluid structure interaction (small deformations of the fluid surface and body), inviscid incompressible, irrotational flow and a linearized Euler-Bernoulli formulation of the fluid pressure. The time-domain analysis of a floating structure involves the calculation of a convolution integral between the impulse response function of the radiation force and the unknown body velocity due to an external force. The convolution integral can be seen as a memory effect where the system response in the past affects the response in the future. Two different time-domain models will be presented. The first one is based on a discretization of the convolution integral. The calculation of the convolution integral is performed at each time step regardless of the chosen numerical scheme. In the second model the convolution integral is replaced by a system of linear ordinary differential equations. The formulation of the state-space model is advantageous regarding the computational effort and the robustness of the solver. Another important feature is the linear-time invariance of the system. In a next step the influence of the nonlinear hydrostatic behavior of the float is investigated by using a simplified formulation.

## 2 Problem formulation

### 2.1 Truncation of the convolution integral

The equation of motion for the analyzed geometry can be formulated by a momentum equilibrium condition around the fixed point, see Fig. 1, which leads to the following equation:

$$(M_{44} + a_{44}^{\infty})\ddot{\varphi}_4(t) + \int_0^t K_{44}(t - \tau)\dot{\varphi}_4(\tau)d\tau + C_{44}\varphi_4(t) + c^{pto}\dot{\varphi}_4(t) = \int_{-\infty}^{\infty} h_4(t - \tau)\eta(t)d\tau \quad (1)$$

Pitch  $\varphi_4(t)$  is the corresponding degree of freedom around the bearing, indicated with the indices  $i = 4, j = 4$ .  $M_{44}$  corresponds to the mass moment of inertia,  $a_{44}^{\infty}$  is the added mass at infinite high frequencies,  $K_{44}(t)$  is the impulse response function of the radiation force,  $C_{44}$  is the hydrostatic stiffness coefficient,  $c^{pto}$  is a constant damping coefficient, representing the linear power take off system,  $h_4(t)$  is the impulse response function of the excitation force and  $\eta(t)$  corresponds to the

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surface elevation. The impulse response function of the radiation force can be seen as the system identity. If we know the response to an impulse, then we know the response to any excitation by convolution with the impulse response function. The basic work for this formulation of the problem was laid by W.E. Cummings (1962) [1]. The convolution integral in Eqn. 1 can be expressed by means of a sum:

$$\int_0^t K_{44}(t-\tau)\dot{\varphi}_4(\tau)d\tau = \Delta t \sum_{\tau=0}^t K_{44}(t-\tau)\dot{\varphi}_4(\tau) \quad (2)$$

Expanding the sum in Eqn. 2, we get the following expression:

$$\Delta t \sum_{\tau=0}^t K_{44}(t-\tau)\dot{\varphi}_4(\tau) = \Delta t [K_{44}(t)\dot{\varphi}_4(0) + K_{44}(t-1)\dot{\varphi}_4(1) + \dots + K_{44}(0)\dot{\varphi}_4(t)] \quad (3)$$

The equation of motion can then be written:

$$(M_{44} + a_{44}^\infty)\ddot{\varphi}_4(t) + K_{44}(0)\dot{\varphi}_4(t) + C_{44}\varphi_4(t) + c^{pto}\dot{\varphi}_4(t) = \int_{-\infty}^{\infty} h_4(t-\tau)\eta(t)d\tau - \int_0^{t^-} K_{44}(t-\tau)\dot{\varphi}_4(\tau)d\tau \quad (4)$$

The numerical integration of Eqn. 4 only requires the calculation of the integral at the preceding time-steps and can therefore be considered as a known quantity. A fourth order Runge Kutta scheme with a constant time step  $\Delta t$  has been used to evaluate the linear equation of motion given in Eqn. 4. Drawbacks of this method are *i)* time consuming *ii)* the convolution integral needs to be calculated at each time step *iii)* the impulse response function needs to be interpolated with the same  $\Delta t$  as the time integration, which is not very convenient. The results are shown in the last page of this abstract. Fairly good agreement can be observed when comparing the numerical discretization of the convolution integral with an analytical calculation for regular waves, i.e. when a constant damping coefficient can be assumed.

## 2.2 Rational approximation to the radiation force

In this section a method is applied to circumvent the drawbacks of the discretization, presented in the previous chapter. The convolution integral is replaced by an equivalent system of coupled first order differential equations, which are solved along with the equations of motion of the absorber, S.R.K Nielsen [2]. The method is based on an initial replacement of the actual frequency response function of the floating body  $H_{r\dot{\varphi}_4}(\omega)$  which was calculated by the software WAMIT, [3]. The approximating rational function is given in the form

$$H_{r\dot{\varphi}_4}(s) \approx \frac{P(s)}{Q(s)} = \frac{p_0s^{m-1} + p_1s^{m-2} + \dots + p_{m-1}s}{s^n + q_1s^{n-1} + \dots + q_n} \quad \left. \vphantom{\frac{P(s)}{Q(s)}}} \right\} \quad s = i\omega \quad (5)$$

The unknowns are the coefficients of polynomials P and Q. The parameters  $p_0, p_1, \dots, p_{m-1}$  and  $q_0, q_1, \dots, q_n$  denotes the *poles* and the *zeros* of the rational approximation and are all real. The order of the filter as given by the pair  $n, m$  may be chosen freely with the only restriction that  $m \leq n$ . A rational causal approximation for  $H_{r\dot{\varphi}_4}$  can be obtained by the MATLAB *control toolbox* [4] or the *MSS FDI toolbox* [5]. Next, the convolution integral is approximated with the product of the constants  $p_0, p_1, \dots, p_{m-1}$  and the new unknowns i.e the additional state vectors  $I(t)$ .

$$\int_0^t K_{44}(t-\tau)\dot{\varphi}_4(\tau)d\tau \approx [ p_0 \ p_1 \ \dots \ p_{m-1} ] I(t) \quad (6)$$

where the time derivation of  $I(t)$  is given as follows:

$$\dot{I}(t) = \begin{bmatrix} -q_1 & -q_2 & -q_3 & q_n \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} I(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\varphi}_4(t) \quad (7)$$

We are now able to approximate the convolution integral of the radiation force by inserting Eqn. 6 into Eqn. 1. As a result we end up in having a time-invariant system for the radiation force which is advantageous regarding computational time and storage requirements.

$$(M_{44} + a_{44}^\infty)\ddot{\varphi}_4(t) + [ p_0 \ p_1 \ \dots \ p_{m-1} ] \dot{I}(t) + C_{44}\dot{\varphi}_4(t) + c^{pto}\dot{\varphi}_4(t) = \int_{-\infty}^{\infty} h_4(t-\tau)\eta(t)d\tau \quad (8)$$

### 2.3 Nonlinear hydrostatic behavior

The change of the hydrostatic pressure at each instantaneous position of the float below the water plane can be characterized by taking into account a nonlinear hydrostatic behavior. This effect can be observed at the two extremities of the red curve, see Fig. 2. On the upper left corner, the float successively dips into the water and on the lower right end of the curve the float starts to be fully submerged by the water. In between, the derivation of the wetted surface is small, hence a linear approximation of the hydrostatic moment becomes justifiable. The red curve is a result of experiments which were carried out at the Hydraulic Laboratory at Aalborg University. In the following model a simplified formulation of the nonlinear hydrostatic effect is presented, where the red curve is approximated by a piecewise trilinear curve, see Fig. 2. The nonlinear force is computed by implementing a displacement control algorithm, i.e. it is assumed that the wave amplitude is zero in the vicinity of the float.

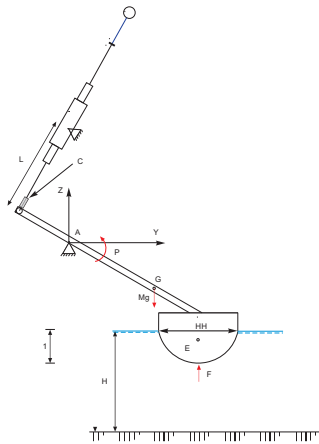


Figure 1: *Wavestar lab model, froude scaled 1:20*

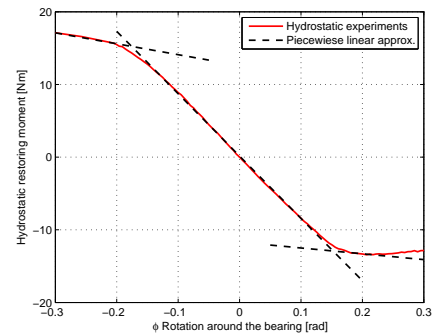


Figure 2: *Hydrostatic restoring moment, piecewise linear approximation*

### 3 Results: Wavestar float - lab model scale 1:20

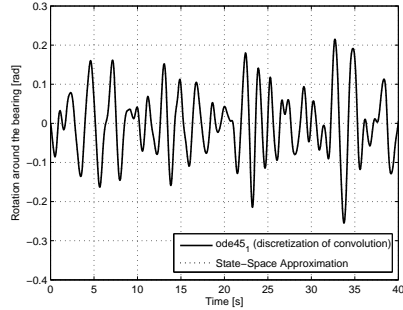


Figure 3: *Body response under panchromatic wave excitation,  $H = 0.1m$ ,  $T = 2.1s$*

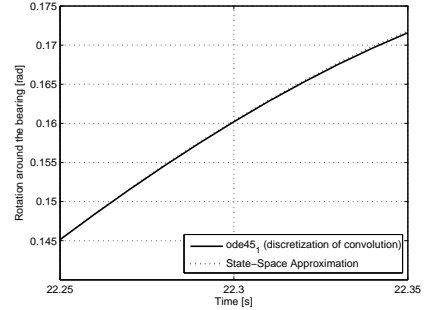


Figure 4: *Body response under panchromatic wave excitation,  $H = 0.1m$ ,  $T = 2.1s$ , zoom*

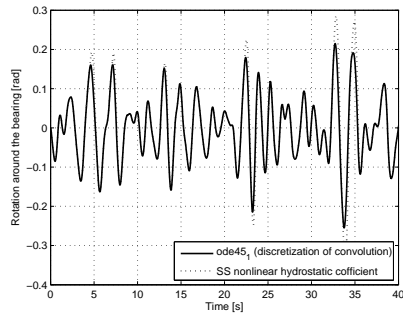


Figure 5: *Non linear hydrostatic, simplified implementation*

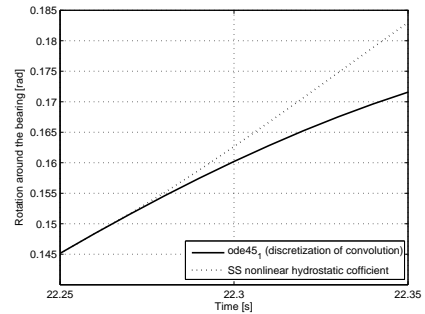


Figure 6: *Non linear hydrostatic, simplified implementation, zoom*

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