Application of the High Level GN Theory to Shallow-Water Wave Problems

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In this work, the high level GN theory is investigated and applied to shallow-water wave problems. The solution algorithm by Demirbilek and Webster (1992) is improved. Two experiments (Zou et al., 2010; Luth et al., 1994) are reproduced numerically. Results of Level 5 GN theory show some difference compared with Level 3 GN theory. Results of Level 5 and Level 7 GN theory are almost the same. For these two cases, results of Level 5 GN theory are the converged GN results. The converged GN results show that the GN theory can predict the wave transformation over uneven seabed accurately. When we use GN theory, the convergence should be considered by using different levels.

1. Introduction

The Green-Naghdi theory, called GN theory for short, was originally developed in 1974 to analyze some nonlinear free-surface flows (Green et al., 1974). In GN models the dimension of a free surface problem is reduced by one, and nonlinear boundary conditions are satisfied on the instantaneous free surface. Demirbilek and Webster (1992) applied Level 2 GN theory to shallow-water wave problems. Ertekin and Kim (1999) studied the hydroelastic behavior of a three-dimensional mat-type VLFS in oblique waves by use of the Level 1 GN theory. Xia et al. (2008) studied the two-dimensional, nonlinear hydroelasticity of a mat-type very large floating structure (VLFS) within the scope of linear beam theory for the structure and the nonlinear, Level 1 GN theory for the fluid. Zhao et al. (2009) applied Level 3 GN theory to shallow-water wave problems. High level (higher than Level 3) GN theory is hard to realize because of the complexity of GN equations. Webster et al. (2011) simplified the GN equations. And, they make the application of high level GN theory easier.

This paper is organized as follows. In section 2, the governing equations for the 2-D GN theory are given. Section 3 gives the improved solution algorithm. Two test cases (Zou et al., 2010; Luth et al., 1994) are reported in section 4. Lastly, the conclusion is given in section 5.

2. GN Theory

The governing equations for the motion of the fluid are provided by the GN theory (Webster et al., 2011). The two-dimensional governing differential equations are

$$\frac{\partial \beta}{\partial t} = \sum_{n=0}^{K} \beta^n \left(w_n - \frac{\partial \beta}{\partial x} u_n \right) \tag{1}$$

$$\frac{\partial}{\partial x}\left(G_n + gS_n\right) + nE_{n-1} - \alpha^n \frac{\partial}{\partial x}\left(G_0 + gS_0\right) = 0 \qquad \text{for} \quad n = 1,2,3,\cdots,K \tag{2}$$

where

$$E_n = \sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t} S_{mn} + \sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x} u_r \right) S_{mrn} + \sum_{r=0}^{K} u_m w_r S_{rn}^{m} \right]$$

$$G_n = \sum_{m=0}^{K}\left[\frac{\partial w_m}{\partial t} S_{mn} + \sum_{r=0}^{K}\left(\frac{\partial w_m}{\partial x} u_r \right) S_{mrn} + \sum_{r=0}^{K} w_m w_r S_{rn}^{m} \right]$$

$$S_n = \int_\alpha^\beta z^n dz\ ,\quad S_{mn} = \int_\alpha^\beta z^{m+n} dz\ ,\quad S_{mrn} = \int_\alpha^\beta z^{m+r+n} dz\ ,\quad S_{rn}^m = m\int_\alpha^\beta z^{m+r+n-1} dz$$

$$u_K = 0$$

$$w_n = -\frac{1}{n}\left(\frac{\partial u_{n-1}}{\partial x}\right) \quad \text{for} \quad n = 1, 2, \cdots, K$$

$$w_0 = -\sum_{n=1}^{K} \alpha^n \left(w_n - \frac{\partial \alpha}{\partial x} u_n \right) + \frac{\partial \alpha}{\partial x} u_0$$

The GN theory with $K = 1$, $K = 2$, $\cdots$ are called Level 1, Level 2, $\cdots$ GN theory, respectively. Zhao et al. (2010) gave the dispersion relations of GN theory with different levels.

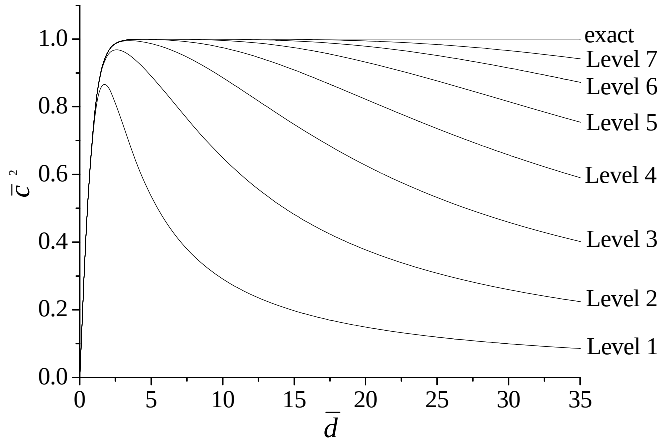


Figure 1: Dispersion relations of the GN theory

3. Algorithm

The solution algorithm by Demirbilek and Webster (1992) is improved here. Eq. (2) can make a system of K coupled, quasi-linear partial differential equations in the K -dependent variables. The variables are expressed here as a K -dimensioned vector, $\xi(x,t)$, whose equations have the special form as

$$\tilde{\mathbf{A}}\dot{\xi}_{,xx} + \tilde{\mathbf{B}}\dot{\xi}_{,x} + \tilde{\mathbf{C}}\dot{\xi} = \tilde{\mathbf{f}} \tag{3}$$

where the dot over ξ signifies a derivative with respect to time, and $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ are $K \times K$ matrices, $\tilde{\mathbf{f}}$ is a K -dimensioned vector.

$$\begin{cases} \dot{\xi} = [\dot{u}_0, \dot{u}_1, \cdots, \dot{u}_{K-1}]^T \\ \dot{\xi}_{,x} = [\dot{u}_{0,x}, \dot{u}_{1,x}, \cdots, \dot{u}_{K-1,x}]^T \\ \dot{\xi}_{,xx} = [\dot{u}_{0,xx}, \dot{u}_{1,xx}, \cdots, \dot{u}_{K-1,xx}]^T \end{cases}$$

$\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{f}}$ are functions of x and ξ and its spatial derivatives, although this dependence will not be shown here in the interest of simplicity. It is assumed that the problem is posed as a two-point boundary-value problem in x and an initial-value problem in t . The domain of x over which a solution to the equations is desired may be assumed to have a uniform grid of x 's, spaced a distance Δx apart. The i-th point on the grid will be denoted by $x_i = i\ \Delta x$, for $i = 1, ns$. Time is also assumed to be discretized with intervals Δt , with $t_j = j\ \Delta t$. The value of the solution vector $\xi(x_i, t_j)$ will be denoted by $\xi^{(i,j)}$ and similar superscripts will be used for the other vectors and matrices.

The spatial derivatives $\dot{\xi}_{,x}$ and $\dot{\xi}_{,xx}$ will be approximated by central differences as

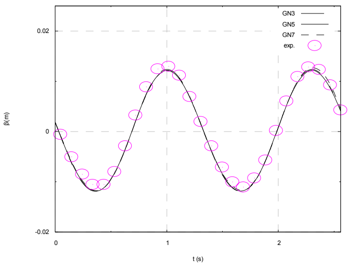
$$
\begin{aligned}
\dot{\xi}_{,x}^{(i,j)} &= (\dot{\xi}^{(i-2,j)} - 8\dot{\xi}^{(i-1,j)} + 8\dot{\xi}^{(i+1,j)} - \dot{\xi}^{(i+2,j)}) / (12\Delta x) \\
\dot{\xi}_{,xx}^{(i,j)} &= (-\dot{\xi}^{(i-2,j)} + 16\dot{\xi}^{(i-1,j)} - 30\dot{\xi}^{(i,j)} + 16\dot{\xi}^{(i+1,j)} - \dot{\xi}^{(i+2,j)}) / 12\Delta x^2
\end{aligned} \tag{4}
$$

With these approximations, Eq. (3) can be written as

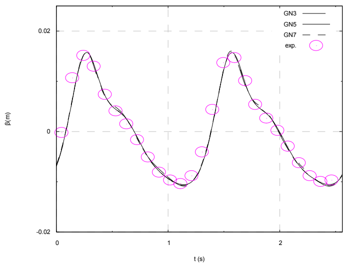
$$
\mathbf{A}^{(i,j)}\dot{\xi}^{(i-2,j)} + \mathbf{B}^{(i,j)}\dot{\xi}^{(i-1,j)} + \mathbf{C}^{(i,j)}\dot{\xi}^{(i,j)} + \mathbf{D}^{(i,j)}\dot{\xi}^{(i+1,j)} + \mathbf{E}^{(i,j)}\dot{\xi}^{(i+2,j)} = \tilde{\mathbf{f}}^{(i,j)} \tag{5}
$$

4. Tests

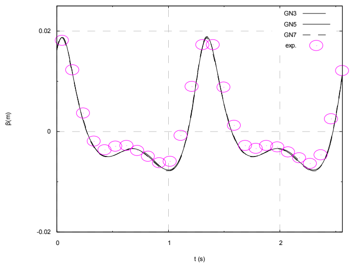
In order to examine the accuracy and capability of the high level GN theory, Level 3, Level 5 and Level 7 GN theory are applied to two examples of 2D wave propagation on uneven bottoms. (1) Zou et al., 2010. (2) Luth et al., 1994.



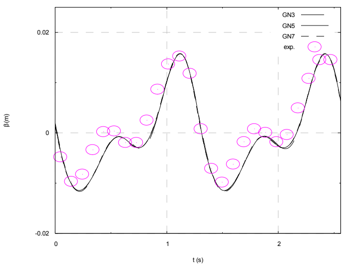
(a) x=7.85m



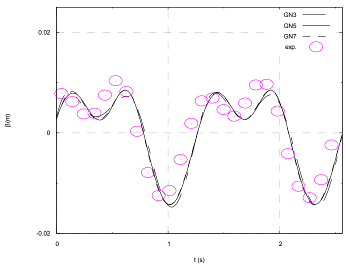
(b) x=8.5m



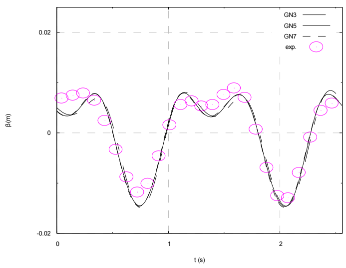
(c) x=9.5m



(d) x=10.5m

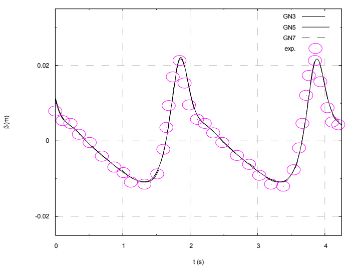


(e) x=13.3m

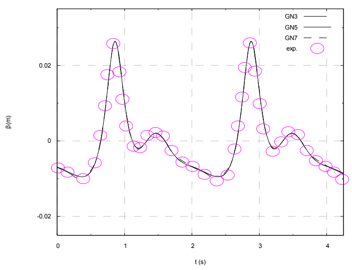


(f) x=15.0m

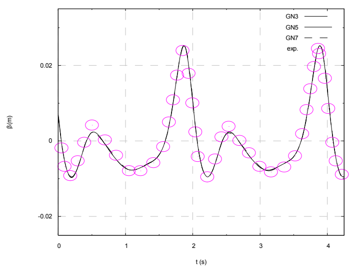
Figure 2: Time series of free surface elevations over a break water with 1:2 front slope and 1:10 back slope (Zou et al., 2010). Wave period is 1.3s. Wave height is 0.02m. (GN3: Dash Dot; GN5: Solid; GN7: Dot)



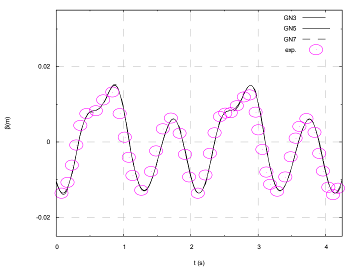
(a) x=12.5m



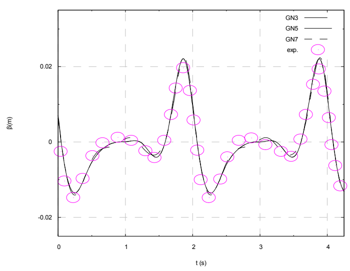
(b) x=13.5m



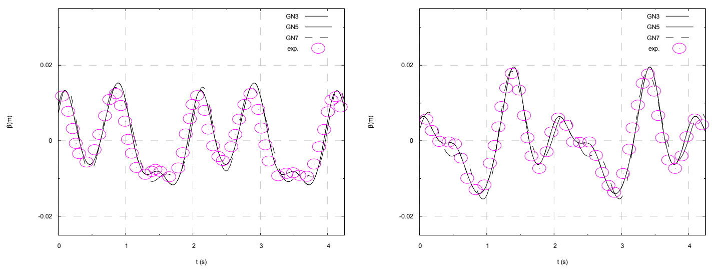
(c) x=14.5m



(d) x=15.7m



(e) x=17.3m



(f) x=19.0m (g) x=21.0m

Figure 3: Time series of free surface elevations for Luth's experiment (1994). Wave period is 2.02s. Wave height is 0.02m. (GN3: Dash Dot; GN5: Solid; GN7: Dot)

5. Conclusions

For the experiment by Zou et al. (2010), results of Level 5 GN theory show some difference compared with Level 3 GN theory at Figure 2(e) and 2(f). For the experiment by Luth et al. (1994), results of Level 5 GN theory show some difference compared with Level 3 GN theory at Figure 3(e), 3(f) and 3(g). Whereas, results of Level 5 and Level 7 GN theory are almost the same for these two experiments. For these two cases, results of Level 5 GN theory are the converged GN results. When we use GN theory, the convergence should be considered by using different levels. The converged GN results show that the GN theory can predict the wave transformation over uneven seabed accurately.

Acknowledgements

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