A non-reciprocal Green's function providing an exact, explicit Dirichlet-Neumann operator: An example for linear waves on a sloping beach in 1DH

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Reciprocity

Like for many other physical phenomena, potential flow problems for water waves follow the reciprocity law: Permuting impulse and response points leaves the response unchanged, although the entire field may look very different. Mathematically, this can be expressed as the source and field point symmetry of the Green's function, i.e. $G(\mathbf{x};\boldsymbol{\xi}) = G(\boldsymbol{\xi};\mathbf{x})$. This property appears for self-adjoint boundary-value problems. The Laplace operator is formally self-adjoint and typical boundary conditions are homogeneous and of the Dirichlet, Neumann, or Robin type ensuring the self-adjointness of the entire boundary value problem. Despite the physical relevance of the reciprocity law for the case considered in the following, it will be shown how a given choice of non-standard boundary conditions for the Green's function gives a non-symmetric result.

Explicit Dirichlet-Neumann operator

The significance of introducing a particular non-symmetric Green's function is that leads to an explicit linear Dirichlet-Neumann operator which maps the velocity potential at the free surface to its normal derivative. This operator provides the kinematic closure needed to compute wave transformation by time-integration of the free-surface boundary conditions. For non-linear waves this linear operator provides the basis for a Dirichlet-Neumann operator expansion as used in the so-called convolution wave model (Shäffer 2005, 2009), which applies a flexible, all-physical-space version of the high-order spectral method (see West et al. 1987, Dommermuth and Yue, 1987, Craig and Sulem, 1993, Bateman et al. 2001, and comparisons by Schäffer, 2008). This abstract considers the linear operator in one horizontal dimension. The vertical free-surface velocity is expressed in terms of the horizontal free-surface velocity, but the trivial step of rewriting results in terms of the velocity potential and extracting the Dirichlet-Neumann operator is omitted.

Convolution method, spectral method

In the following (u,w) is the fluid velocity in a Cartesian coordinate system (x,z) with z pointing upwards and $z=0$ at the still-water level. Subscript zero refers to variables at $z=0$ except for the water depth, h, where it signifies constant depth. For mild bottom slopes, the kinematic closure may be given in terms of the convolution integral

$$w_0(x) = \int_a^b u_0(x')r(x,x')dx' \tag{1}$$

where the impulse response function is

$$r(x,x') = -\frac{1}{2h(x)}\operatorname{csch}\left(\frac{\pi}{2}\int_x^{x'}\frac{1}{h(x'')}dx''\right) \tag{2}$$

for a mildly sloping bottom. The limits of integration are in principle $(a,b)=(-\infty,\infty)$, but for practical computation the exponential decay of the impulse response function allows for truncation just several water depths away from the observation point, x. Lateral boundaries may also limit the range of integration, where reflective conditions may be incorporated by imaging the impulse response function.

For a constant, small slope, h_x, where $h(x) = h_x x$, (2) reduces to

$$r(x,x') = -\frac{1}{2h_x x}\operatorname{csch}\left(\frac{\pi}{2h_x}\log\left(\frac{x'}{x}\right)\right) \tag{3}$$

while the relevant limits become $(a,b)=(0,\infty)$, and for constant depth the impulse response function is

$$r(x,x') = -\frac{1}{2h_0}\operatorname{csch}\left(\frac{\pi}{2}\frac{x'-x}{h_0}\right) \tag{4}$$

The mild-slope result was derived by Matsuno (1993) using conformal mapping and by Schäffer (2005, 2009) using infinite series differential operators.

For constant depth, the Dirichlet-Neumann operator may be applied through multiplication by the transfer function $-k\tanh kh_0$ in wavenumber space. This may be seen by introducing the velocity potential and (4) in (1) and applying the convolution theorem.

Relation to boundary integral equations

Although the convolution integral (1) is quite simple, the underlying derivation was rather complicated and further generalization towards variable 2DH bathymetry and complex-shaped domains could benefit from a new framework of derivation. Observing that (1) is merely a boundary integral, it is relevant to investigate the relation to integral theorems. While the convolution approach provides an explicit expression for the quantity needed in the kinematics closure, it is recalled that usual boundary integral equation methods for water waves are implicit and involve the inversion of dense linear systems. Is it possible to use the versatility of boundary integral theorems to further generalize the explicit convolution method?

Scalar Green's 2nd identity

Green's 2nd identity is commonly applied for the velocity potential and a Green's function to obtain

$$\alpha(x,z)\phi(x,z) = \int_{\Gamma}\left[\phi(x',z')G_{n'}(x,z;x',z') - G(x,z;x',z')\phi_{n'}(x',z')\right]d\Gamma \tag{5}$$

where (x',z') belongs to the boundary, Γ, subscript n' indicates a derivative in the outward normal direction, $\alpha(x,z)$ is zero for points external to the fluid, 2π for internal points, and equals the interior angle (typically π) for observation points on Γ. Since $w=\phi_z$ satisfies the Laplace equation, the BIE is equally valid when substituting ϕ by w and further evaluating this at $z=0$ gives

$$w_0(x) = \frac{1}{\alpha(x,0)}\int_{\Gamma}\left[w(x',z')G_{n'}(x,0;x',z') - G(x,0;x',z')w_{n'}(x',z')\right]d\Gamma \tag{6}$$

To match the convolution method for constant depth, $h=h_0$, consider the Green's function constituted by an infinite array of vertically aligned alternating sources and sinks with a distance of $2h_0$ between consecutive singularities. This definition of the Green's function is relevant for both 1DH and 2DH. Although the series for the 1DH case is divergent, it does have a generalized limit that may be found analytically as

$$G(x,z;x',z') = \mathrm{Re}\left[\log\left(\tanh\left(\frac{\pi}{4}\frac{(x-x')+\mathrm{i}(z-z')}{h_0}\right)\right)\right] \tag{7}$$

This Green's function decays exponentially in the horizontal direction. Consequently, the influence of lateral boundaries vanishes just several water depths away from the observation point. While reflective lateral boundaries can be accounted for by the image method, we choose to ignore them in following by which (6) reads

$$\begin{aligned} w_0(x) = &\frac{1}{\pi}\int_{\Gamma_0}\left[w_0(x')G_{z'}(x,0;x',0) - G(x,0;x',0)w_{z'}(x',0)\right]d\Gamma_0 + \\ &\frac{1}{\pi}\int_{\Gamma_b}\left[w(x',z')G_{n'}(x,0;x',z') - G(x,0;x',z')w_{n'}(x',z')\right]d\Gamma_b \end{aligned} \tag{8}$$

where Γ_0 is the surface at $z'=0$ and Γ_b is the bottom at $z'=-h(x')$. For constant depth, $h=h_0$, both terms in the bottom integral vanish, the first one due to the bottom boundary condition for the vertical velocity and the second one due to the vanishing Green's function. Furthermore, the first term in the integration over Γ_0 vanishes, since $G_{z'}(x,0;x',0)=0$. Letting Γ_0 extend over the entire horizontal axis, the remainder of (8) is

$$w_0(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} G(x,0;x',0)w_{z'}(x',0)dx' \tag{9}$$

By continuity, $w_{z'}=-u_{x'}$, followed by integration by parts, this becomes

$$w_0(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} G_{x'}(x,0;x',0)u_0(x')dx' \tag{10}$$

From (7), we get

$$-\frac{1}{\pi}G_{x'}(x,0;x',0) = -\frac{1}{2h_0}\operatorname{csch}\left(\frac{\pi}{2}\frac{x'-x}{h_0}\right) \tag{11}$$

which substituted in (10) recovers the result of the convolution method in (4).

While the choice of using w instead of ϕ in Green's 2nd identity is successful in providing an explicit result for constant depth, it is not clear how this approach could be used for variable depth, where the bottom boundary condition generalizes to $\phi_{n'} = 0$ rather than $w = 0$. The bottom boundary condition no longer annihilates one term leaving a condition on the Green's function to annihilate the other. This problem calls for an alternative approach and one option is as follows.

Vector Green's 2nd identity

Vector Greens 2nd identity (see Morse and Feshbach, 1953, p 1768) was used by Nwogu (2009) to study waves on sheared currents. Following Nwogu (2009, eq. 3.7), but assuming potential flow, provides an expression for the normal velocity in terms of the normal and tangential velocities at the boundary. For an observation point at $z = 0$, this reads in 1DH

$$\begin{aligned} w_0(x) = &\frac{1}{\pi}\int_{\Gamma_0}\left[w_0(x')G_{z'}(x,0;x',0) - u_0(x')G_{x'}(x,0;x',0)\right]d\Gamma_0 + \\ &\frac{1}{\pi}\int_{\Gamma_b}\left[v_N(x',z')G_{z'}(x,0;x',z') + v_T(x',z')G_{x'}(x,0;x',z')\right]d\Gamma_b \end{aligned} \tag{12}$$

where v_N and v_T are the normal and tangential velocities

$$(v_N, v_T) = \left((u,w)\cdot(n'_x, n'_z), (u,w)\cdot(-n'_z, n'_x)\right) \tag{13}$$

Note that the directions of the Green's function derivatives are linked to the 'observation direction' and thus remain vertical and horizontal even for variable depth.

For constant depth, (12) reduces to (10) and thus the vector form of Green's 2nd identity provides the same result as the scalar form of Green's 2nd identity used above. This confirms the equivalence with the convolution expression (4). The advantage of (12) appears for variable depth, where, as opposed to (8), the bottom boundary condition now annihilates the first term in the boundary integral. To get a formulation like the convolution integral in (1), which is independent of bottom velocities, the last term in (12) must also vanish. This requires that $G_{x'}(x,0;x',z') = 0$ on the bottom i.e. a non-physical boundary condition. To keep the expression for w_0 explicit, the property $G_{z'}(x,0;x',0) = 0$ must be retained, but this is easily done just by choosing $G(x,0;x',z')$ as an even function of z'.

Symmetric Green's function; constant bottom-slope

As an intermediate step towards a Green's function that satisfies the above conditions, regard the symmetric Green's function

$$G(x,z;x',z') = \operatorname{Re}\left[\sum_{n=0}^{2N-1}(-1)^n \log\left(\frac{x+\mathrm{i}z}{x'+\mathrm{i}z'} - \exp\left(2\pi\mathrm{i}\frac{n}{2N}\right)\right)\right] = \operatorname{Re}\left[\log\left(\frac{1-\left(\frac{x+\mathrm{i}z}{x'+\mathrm{i}z'}\right)^N}{1+\left(\frac{x+\mathrm{i}z}{x'+\mathrm{i}z'}\right)^N}\right)\right] \tag{14}$$

as relevant for $h(x) = h_x x$ with $h_x = \tan\mu$ and $\mu = \pi/(2N)$ where N is integer. Like (7), this is constituted by equidistant, alternating sources and sinks, but now distributed along a circle of radius x centred at $x' = 0$. In a suitable limit of vanishing bottom slope (14) reduces to (7). Inspection shows that (14) has $G_{x'}(x,0;x',z') \neq 0$ on the bottom and thus it does not satisfy the required conditions. However, (14) does have $G_x(x,0;x',z') = 0$ on the bottom, where the derivative is now taken with respect to the field point abscissa instead of the integration point abscissa. Although this is not the desired property, it does give a hint on how to proceed.

Non-symmetric Green's function and explicit Dirichlet-Neumann operator; constant bottom-slope

Although the required property only relates to the bottom, let us look for a function Γ that satisfies the general condition

$$\Gamma_{x'}(x,z;x',z') = -G_x(x,z;x',z') \tag{15}$$

where G is defined in (14). A solution is

$$\Gamma(x,z;x',z') = \mathrm{Re}\left[\sum_{n=0}^{2N-1}\exp\left(-2\pi\,\mathrm{i}\frac{n}{2N}\right)(-1)^n\log\left(\frac{x+\mathrm{i}\,z}{x'+\mathrm{i}\,z'}-\exp\left(2\pi\,\mathrm{i}\frac{n}{2N}\right)\right)\right] \tag{16}$$

which is similar to (14) except for each complex component being rotated by $-\pi n/N$ in the complex plane. By inspection, Γ turns out to be a suitable Green's function that satisfies $\Gamma_{x'}(x,0;x',z')=0$ on the bottom while also satisfying $\Gamma_{z'}(x,0;x',0)=0$. This Green's function is not symmetric, since in general we have $\Gamma(x,z;x',z') \neq \Gamma(x',z';x,z)$. Thus Γ is not invariant to permutation of impulse and response despite the fact that the physical problem in question does have this property.

Using Γ as Green's function in (12) while looking at the case of constant bottom slope, what remains is

$$w_0(x) = -\frac{1}{\pi}\int_0^\infty \Gamma_{x'}(x,0;x',0)u_0(x')dx' \tag{17}$$

or

$$w_0(x) = \frac{1}{\pi}\int_0^\infty G_x(x,0;x',0)u_0(x')dx' \tag{18}$$

This expression is exact. Further simplification yields

$$-\Gamma_{x'}(x,z;x',z') = G_x(x,z;x',z') = \mathrm{Re}\left[\frac{N}{x+\mathrm{i}\,z}\mathrm{csch}\left(N\log\left(\frac{x+\mathrm{i}\,z}{x'+\mathrm{i}\,z'}\right)\right)\right] \tag{19}$$

by which,

$$-\frac{1}{\pi}\Gamma_{x'}(x,0;x',0) = -\frac{1}{2\mu x}\mathrm{csch}\left(\frac{\pi}{2\mu}\log\left(\frac{x'}{x}\right)\right) \tag{20}$$

For mild slopes, $\mu = \arctan(h_x)$ may be replaced by h_x and the mild-slope result in (3) is recovered.

Although focus has been on a specific example, it appears that the result (18), while noting that the derivative is taken with respect to the field point and not the integration point, is valid in general, if G is a Green's function that for observation points at the surface, $z=0$, satisfies homogeneous Dirichlet boundary conditions on the bottom and homogeneous Neumann conditions on the surface, $z'=0$.

Concluding remarks

Green's functions relevant for physical problems obeying the reciprocity law (invariance to source-receiver permutation) usually display field-point integration-point symmetry. Yet, it has been shown for such a physical problem how a non-symmetric Green's function obeying non-physical boundary conditions can be useful in the derivation of an explicit Dirichlet-Neumann operator viz. the exact solution for 1DH linear waves on a constant slope of inclination $\pi/(2N)$.

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