

Effective treatment of Fourier integrals associated with a hemi-sphere advancing in waves

I. Ten and X.B. Chen

Research Department, BV, 92571 Neuilly-Sur-Seine, France

igor.ten@bureauveritas.com

Unlike the traditional way to evaluate potential flows by integrating a singularity distribution on the hull with the associated Green function resultant from a Fourier integral, the order to perform the Fourier integral before the space integral (on the hull) is inverted following the approach proposed by Kochin and given in [1]. The Fourier integral then involves a spectral function depending on the singularity distribution. To treat effectively the Fourier integral on (k, θ) plane, the order to perform the k -integral before the θ -integral is again inverted following the approach presented in [2] since the θ -integral can be expressed analytically by using the Cauchy theorem of residues. This leads to a final form of velocity potential by a *single* integral in k which can be performed numerically using an appropriate and efficient algorithm.

Introduction

In [3], it is proposed to separate the fluid domain into two sub-domains and solve the problem in each sub-domain by the specific method: the Rankine source method for the internal domain and the Kelvin source method for the external domain. On the interface, two solutions must be matched. The control surface, which divides the full fluid domain is proposed to be hemi-spheroid, in order to reduce the number of panels on the free-surface for the Rankine source method. In the current work we propose to replace the hemi-spheroid by the hemi-sphere, which is the particular case, and find the spectral function for the hemi-sphere with forward speed.

For the convenience, the spherical coordinate system (c, φ, β) is defined as the following

$$x = c \sin \beta \cos \varphi, \quad y = c \sin \beta \sin \varphi, \quad z = -c \cos \beta, \quad (1)$$

where $\varphi \in [-\pi, \pi]$ varies in the plane xOy , $\beta \in [0, \pi/2]$ - in the plane yOz , and $c \in [0, \infty)$ is radius of the sphere and the Cartesian reference system (x, y, z) , moving with the hemi-sphere of radius c at the mean forward speed U along the positive x -axis, is defined by letting (x, y) plane coincide with the mean free surface and z -axis be positive upward. It is assumed that the fluid is ideal and the flow is irrotational, the wave steepness is small and the depth is infinite.

The sphere is presented by distribution of singularities for which the Green function is well known and is given as the following, see [4] and [5]

$$4\pi G(P, Q) = -\frac{1}{r(P, Q)} + \frac{1}{r^*(P, Q)} + \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{k(z+\zeta) - i(k_\alpha(x-\xi) + k_\beta(y-\eta))}}{D + i\epsilon D_f} dk_\alpha dk_\beta, \quad (2)$$

where $k = \sqrt{k_\alpha^2 + k_\beta^2}$, $P = (x, y, z)$ is a field point, $Q = (\xi, \eta, \zeta)$ is a source point, $r(P, Q)$ and $r^*(P, Q)$ are the distances between the field and the source points and between the field point and the mirror of the source point with respect to the mean free surface $z = 0$, $D = (Fk_\alpha - f)^2 - k^2$, $f = \omega \sqrt{L/g}$ is nondimensional frequency, $F = U/\sqrt{gL}$ is the Froude number, $\tau = fF$ is the Brard number.

From the theory of orthogonal functions the velocity potential $\phi(x, y, z)$ and its normal derivative $\psi(x, y, z) \equiv \phi_n(x, y, z)$ may be present in the form of infinite series with respect to the Legendre functions $P_n(t)$ and the exponential functions $e^{il\varphi}$:

$$\phi(x, y, z) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \phi_{kl} p_k(t) e^{il\varphi}, \quad \psi(x, y, z) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \psi_{kl} p_k(t) e^{il\varphi}, \quad (3)$$

where $t = 2 \cos \beta - 1$, $p_k(t)$ are the normalized Legendre functions. With the help of the Green second identity

$$\sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \psi_{kl} \left(\frac{c^2}{4\pi} A_{nm,kl} + B_{nm,kl} \right) = \frac{1}{2} \phi_{nm} + \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \phi_{kl} \left(\frac{c^2}{4\pi} C_{nm,kl} + D_{nm,kl} \right). \quad (4)$$

the coefficients ψ_{kl} are expressed through ϕ_{kl} , which are assumed to be known, as a solution of the internal domain.

The coefficients $A_{nm,kl}$ and $C_{nm,kl}$ are associated with the integrals over the hemi-sphere, and $B_{nm,kl}$ and $D_{nm,kl}$ - alone the water line. Particularly, the coefficients $A_{nm,kl}$ and $C_{nm,kl}$ are

$$\begin{aligned} A_{nm,kl} &= \int_{-1}^1 \int_{-\pi}^{\pi} \int_{-1}^1 \int_{-\pi}^{\pi} p_k(t') e^{il\varphi'} p_n(t) e^{-im\varphi} G(P, Q) d\varphi' dt' d\varphi dt, \\ C_{nm,kl} &= \int_{-1}^1 \int_{-\pi}^{\pi} \int_{-1}^1 \int_{-\pi}^{\pi} p_k(t') e^{il\varphi'} p_n(t) e^{-im\varphi} G_n(P, Q) d\varphi' dt' d\varphi dt. \end{aligned} \quad (5)$$

The coefficients $B_{nm,kl}$ and $D_{nm,kl}$ are similar to $A_{nm,kl}$ and $C_{nm,kl}$ and they are treated in the same manner.

Hereafter, we consider only the free-surface component represented by the Fourier integral in (2) of the Green function. Moreover, we are going to deal only with $A_{nm,kl}$ and $C_{nm,kl}$ in (4) because the same technique is applicable for the rest two coefficients.

The substitution of the wave component of the Green function $G(P, Q)$ and $G_n(P, Q)$ into (5), and the integration of the results with respect to φ and φ' provide in the polar coordinate system (k, θ) , so that $k_\alpha = k \cos \theta$ and $k_\beta = k \sin \theta$,

$$\begin{aligned} A_{nm,kl} &= i^{l-m} F^2 \int_0^\infty k \mathcal{I}_{nm}(kc) \mathcal{I}_{kl}(kc) g_{l-m}(k) dk, \\ C_{nm,kl} &= i^{l-m} F^2 \int_0^\infty k^2 \mathcal{I}_{nm}(kc) \mathcal{I}'_{kl}(kc) g_{l-m}(k) dk, \end{aligned} \quad (6)$$

where

$$\mathcal{I}_{nm}(k) \equiv \int_{-1}^1 p_n(t) J_m \left(k \frac{\sqrt{4 - (t+1)^2}}{2} \right) e^{-k \frac{t+1}{2}} dt \quad (7)$$

and

$$g_n(k) \equiv \lim_{\epsilon \rightarrow 0^+} \frac{1}{F^2} \int_{-\pi}^{\pi} \frac{e^{in\theta}}{D + i\epsilon D_f} d\theta \quad (8)$$

with $J_m(\cdot)$ is the Bessel function of the first kind. In (6), $\mathcal{I}'_{kl}(k) = d\mathcal{I}_{kl}(k)/dk$. The functions $\mathcal{I}_{nm}(k)$ and their derivatives do not have any difficulties in numerical calculations of the integrals. In the special case of steady flow ($f = 0$), similar θ -integral was analysed in [6].

The function $g_n(k)$.

The main difficulties resides in the evaluation of the function $g_n(k)$ given by (8). After the substitution the definition of the D into (8) and changing the variable $k_F = F^2 k$, the function $g_n(k)$ becomes

$$g_n(k) = \lim_{\epsilon \rightarrow 0^+} g_n^\epsilon(k), \quad g_n^\epsilon(k) \equiv F^2 \int_{-\pi}^{\pi} \frac{e^{in\theta}}{(k_F \cos \theta - \tau)^2 - k_F - 2i\epsilon(k_F \cos \theta - \tau)} d\theta. \quad (9)$$

Due to $g_n^\epsilon(k) = g_{-n}^\epsilon(k)$ we assume that $n \geq 0$. The variable change $z = e^{i\theta}$ ($z d\theta = -idz$) leads (9) to

$$g_n^\epsilon(k) = -\frac{4i}{k_F^2} \int_C \frac{z^{n+1} dz}{(z^2 - 2a_-^\epsilon + 1)(z^2 - 2a_+^\epsilon + 1)}; \quad a_\pm^\epsilon = \frac{\tau \pm \sqrt{k_F} \sqrt{1 - \epsilon^2} + i\epsilon}{k_F}, \quad (10)$$

where C is a circle of the unit radius. The denominator in (10) has 4 simple zeros:

$$\begin{aligned} z_{1,2}^\epsilon &= a_+^\epsilon \mp \sqrt{(a_+^\epsilon)^2 - 1} \xrightarrow{\epsilon \rightarrow 0^+} z_{1,2} = \begin{cases} a_+ \mp i\sqrt{1 - a_+^2} & \text{for } 0 < a_+ < 1; \\ a_+ \mp \sqrt{a_+^2 - 1} & \text{for } 1 < a_+; \end{cases} \\ z_{3,4}^\epsilon &= a_-^\epsilon \mp \sqrt{(a_-^\epsilon)^2 - 1} \xrightarrow{\epsilon \rightarrow 0^+} z_{3,4} = \begin{cases} a_- \pm i\sqrt{1 - a_-^2} & \text{for } -1 < a_- < 0; \\ a_- \mp i\sqrt{1 - a_-^2} & \text{for } 0 < a_- < 1; \\ a_- \mp \sqrt{a_-^2 - 1} & \text{for } 1 < a_-^2. \end{cases} \end{aligned} \quad (11)$$

In Fig. 1 it is shown the positions of the poles depending on the parameters k and τ . For a fixed $\tau < 1/4$ we may define the following 6 intervals for k : $I_1 = (0, k_4)$ and $I_2 = (k_4, \infty)$, $I_3 = (0, k_1)$, $I_4 = (k_1, k = \tau)$,

$I_5 = (k = \tau, k_2) \cup (k_3, \infty)$, and $I_6 = (k_2, k_3)$, where

$$k_1 = \frac{(-1 + \sqrt{1 + 4\tau})^2}{4F^2}, \quad k_{2,3} = \frac{(1 \mp \sqrt{1 - 4\tau})^2}{4F^2}, \quad k_4 = \frac{(1 + \sqrt{1 + 4\tau})^2}{4F^2}, \quad (12)$$

and for $\tau > 1/4$ - the first 4 intervals, I_{1-4} , with modified $I_5 = (k = \tau, \infty)$.

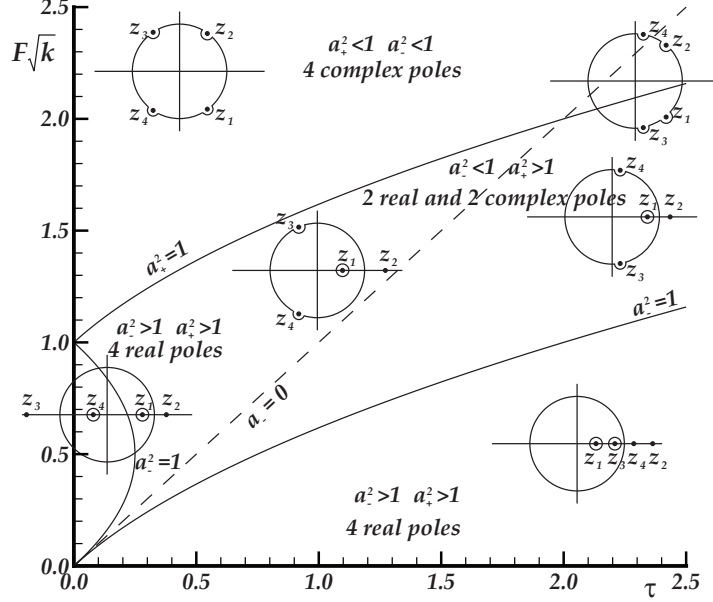


Figure 1: Complex zeros of the denominator $D + i\epsilon D_f$ in $(\tau, \sqrt{k_F})$ plane.

The analysis shows that the real part of $z_{3,4}^\epsilon$ is always positive, and it can be shown that $\|z_3^\epsilon\| < 1 < \|z_4^\epsilon\|$. In addition, the behaviour of the poles $z_{1,2}^\epsilon$ with respect to ϵ depending on the k is as the following:

- 1a) $k \in I_1$. The poles $z_{1,2}^\epsilon$ tend to the real values $z_{1,2}$ so that $\text{Im}z_1^\epsilon < 0$, $\text{Im}z_2^\epsilon > 0$.
- 1b) $k \in I_2$. The poles $z_{1,2}^\epsilon$ tend to the complex values $z_{1,2}$, where $\|z_{1,2}\| = 1$, $\text{Im}z_1^\epsilon < 0$, $\text{Im}z_2^\epsilon > 0$.

On the other hand, the moduli of $z_{3,4}^\epsilon$ are dependent on the real part of the pole.

- 2a) $k \in I_3$. The poles $z_{3,4}^\epsilon$ tend to the real values $z_{3,4}$ so that $\text{Im}z_3^\epsilon < 0$, $\text{Im}z_4^\epsilon > 0$ and $\|z_1^\epsilon\| < \|z_3^\epsilon\| < 1 < \|z_4^\epsilon\| < \|z_2^\epsilon\|$.
- 2b) $k \in I_4$. $z_{3,4}$ are complex and $\|z_{3,4}\| = 1$, $\text{Im}z_3 < 0$, $\text{Im}z_4 > 0$. The poles $z_{3,4}^\epsilon$ have the following properties $\|z_3^\epsilon\| < 1$, $\|z_4^\epsilon\| > 1$; $\text{Re}z_{3,4}^\epsilon > 0$.
- 2c) $k \in I_5$. $z_{3,4}$ are complex with unit moduli as in 2b), but $\text{Im}z_3 > 0$, $\text{Im}z_4 < 0$. The real part of the poles $z_{3,4}^\epsilon$ is negative and $\|z_4^\epsilon\| < 1 < \|z_3^\epsilon\|$.
- 2d) $k \in I_6$. This case is similar to the case 2a), but $\text{Im}z_3^\epsilon > 0$, $\text{Im}z_4^\epsilon < 0$ and $\text{Re}z_3^\epsilon < -1 < \text{Re}z_4^\epsilon < 0 < \text{Re}z_1^\epsilon < 1 < \text{Re}z_2^\epsilon$.

The application of the residue theorem to the integral in (10) and, then, taking the limit $\epsilon = 0$ yield

$$g_n(k) = \frac{8\pi}{k_F^2} \left(\text{res}_{z=z_1} \frac{z^{n+1}}{(z^2 - 2a_+z + 1)(z^2 - 2a_-z + 1)} + \text{res}_{z=z_*} \frac{z^{n+1}}{(z^2 - 2a_+z + 1)(z^2 - 2a_-z + 1)} \right), \quad (13)$$

where $z_* = z_3$ if $a_- > 0$ and $z_* = z_4$ otherwise. Evaluating the residues in (13) provides

$$g_n(k) = \frac{2\pi}{k_F \sqrt{k_F}} \left\{ \frac{z_1^n}{z_1 - z_2} - \text{sign}(a_-) \frac{z_*^n}{z_3 - z_4} \right\} \quad (14)$$

with z_{1-4} given by (11). For $\tau \neq 0$, we have $a_+ \approx a_- \approx a_0 = \tau/k_F \gg 1$ for $k \rightarrow 0$ then

$$g_n(k) \approx \frac{\pi}{F\sqrt{k}} \left(a_0 - \sqrt{a_0^2 - 1} \right)^n \left(\frac{1}{\tau - F\sqrt{k}} - \frac{1}{\tau + F\sqrt{k}} \right) = \frac{2\pi}{\tau^2} \left(a_0 - \sqrt{a_0^2 - 1} \right)^n = \frac{2\pi}{\tau^2} \left(\frac{F^2 k}{2\tau} \right)^n \quad (15)$$

since $a_0 - \sqrt{a_0^2 - 1} \approx 1/(2a_0) = F^2 k/(2\tau)$. On the other side for $k \rightarrow \infty$, we have $a_\pm \approx \pm 1/(F\sqrt{k})$ and $(1 - a_\pm^2)^{\pm \frac{1}{2}} \approx 1 \mp 0.5a_\pm^2 \approx 1$. thus :

$$g_n(k) \approx (-i)^{n-1} 2\pi\tau / (F^6 k^3) \quad (16)$$

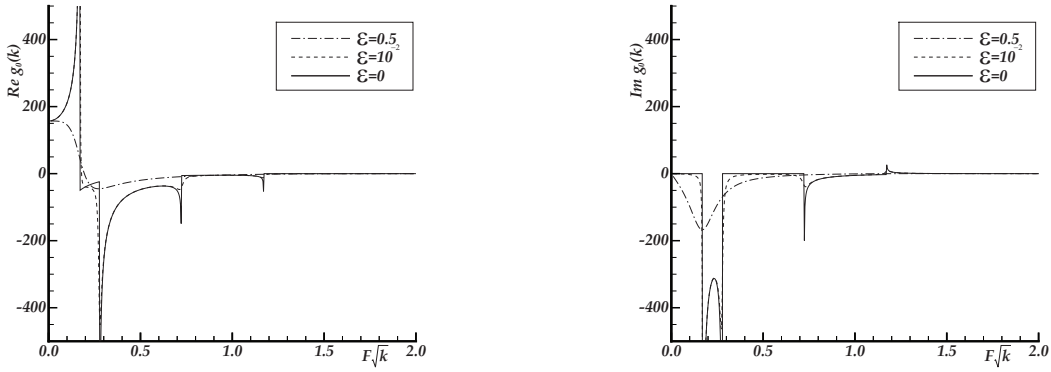


Figure 2: Real (left) and imaginary (right) parts of $g_0^\epsilon(k)$ for $\tau = 0.2$, $F = 1$ and different epsilon: $\epsilon = 0.5$ (dash-dotted), $\epsilon = 0.01$ (dashed) and $\epsilon = 0$ (solid).

showing that $g_n(k)$ decreases rapidly as $k \rightarrow \infty$ in order of $O(k^{-3})$.

Discussions

An example of variations of $g_n^\epsilon(k)$ to $g_n(k)$ is shown on Fig. 2 for $n = 0$; $\epsilon = 0$ (solid), 0.5 (dash-dotted), and 0.01 (dashed); $\tau = 0.2$ and $F = 1$. The real part of the function is depicted on the left while the imaginary part on the right. The function $g_n^\epsilon(k)$ has sharp variation but smooth for $\epsilon > 0$ when k crosses the points (k_1, k_2, k_3, k_4) for $\tau < 1/4$ or (k_1, k_4) for $\tau > 1/4$. At $\epsilon = 0$, the function $g_n(k)$ behaves like $O(1/\sqrt{k - k_i})$ with $i = 1, 2, 3, 4$ for $\tau < 1/4$ and $i = 1, 4$ for $\tau > 1/4$. Numerical integrations can be performed by extracting the singularities which are integrated analytically.

The single representation (6) is possible as far as the spectral function of (k, θ) involving the analytical expression of θ . This is probably the case for any distribution of singularities. For unit spectral function corresponding to a point source, we have the free-surface component of the Green function which can be transformed to

$$G(P; Q) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} (-i)^n e^{-in\gamma} \int_0^{\infty} k e^{k(z+\zeta)} J_n(kR) g_n(k) dk, \quad (17)$$

by using the Jacobi-Anger expansion: $e^{ikR \cos(\theta-\gamma)} = \sum_{n=-\infty}^{\infty} i^n J_n(kR) e^{in(\theta-\gamma)}$. Thus, the present work on $g_n(k)$ is useful to develop new formulation of Green functions based on the series (17).

References

- [1] F. Noblesse, X.B. Chen X.B. and C. Yang, 1997, "Fourier-Kochin theory of free-surface flows", Proc. 21st Symp Naval Hydro. pp.120-135.
- [2] T. Nguyen and X.B. Chen, 2003, "A Single-Integral Representation for the Green Function of Steady Ship Flow in Water of Finite Depth", Proc. 18th IWWF, Le Croisic, France.
- [3] I. Ten and X.B. Chen, 2010, "A coupled Rankine-Green function method applied to the forward-speed seakeeping problem", Proc. 25th IWWF, Harbin, China, pp.165-168.
- [4] X.B. Chen, F. Dias and W.Y. Duan, 2011. "Introduction of dissipation in potential flows", Proc. of the Seventh International Workshop on Ship Hydrodynamics, Shanghai, China, pp.213-222.
- [5] X.B. Chen and D.Q. Lu, 2009, "Time-harmonic ship waves with the effect of surface tension and fluid viscosity", Proc. 24th IWWF, Saint-Petersburg, Russia, pp.46-49
- [6] Diebold, L., 2007. "Study of the Neumann-Kelvin problem for one hemisphere", Proc. 22nd IWWF, Plitvice, Croatia, pp.57-60