GENERATION OF UNSTEADY WAVES BY THREE-DIMENSIONAL SOURCE IN DEEP WATER WITH AN ELASTIC COVER

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1. Introduction

The velocity potential of a transient source of arbitrary strength and in arbitrary three-dimensional motion is derived, wherein a thin elastic plate of infinite extent is assumed to cover the surface of water. This potential is fundamental to the analysis of various types of body motion in deep water under the influence of waves. As a sample application, the potential of a time-harmonic source with forward speed is obtained from the transient source by specifying the appropriate source strength and motion.

Wave motion due to fundamental line and point singularities with time-dependent strength submerged in water with an elastic cover and an inertial surface was investigated previously by Chowdhury & Mandal (2006), Lu & Dai (2006, 2008a, 2008b). Basically, two kinds of unsteadiness were considered, namely, instantaneous or time-harmonic singularities.

2. Mathematical formulation

Consider a fixed, rectangular coordinate system O-xyz where (xy)-plane coincides with the undisturbed upper surface of water, and the positive z-axis points upwards. The initially quiescent fluid of infinite depth is assumed to be inviscid, incompressible and homogeneous. Upper surface is covered by a thin layer of elastic material of uniform density with the lateral stress. The motion in the fluid is generated due to a point mass-source of time-dependent strength which starts operating at time $t = 0$. The linearized initial-boundary-value problem is formulated within the framework of potential flow. Position of a source and its strength at time $t \geq 0$ are given by $\vec{\xi}(t) = (\xi(t), \eta(t), \zeta(t))$ and $\mu(t)$, where $\zeta(t) < 0$ and $\mu(t) = 0$ for $t < 0$.

The motion in the fluid can be described by a velocity potential $\Phi(\vec{x}, t)$ $(t > 0)$. In the linear theory, Φ satisfies in the fluid domain

$$\Delta\Phi = \mu(t)\delta(\vec{x} - \vec{\xi}(t), \qquad (1)$$

where Δ denotes the three-dimensional Laplace operator, $\vec{x} = (x, y, z)$, δ is the Dirac delta function.

If $w(x, y, t)$ denotes the small vertical displacement of the upper surface below its equilibrium position, then the linearized kinematic and dynamic conditions at the upper surface are given by

$$\partial w/\partial t = \partial\Phi/\partial z,$$
$$D\Delta_2^2 w + Q\Delta_2 w + M\partial^2 w/\partial t^2 +$$
$$\rho\partial\Phi/\partial t + g\rho w = 0 \quad (z = 0), \qquad (2)$$

where

$$D = Eh_1^3/[12(1-\nu^2)], \;\; M = \rho_1 h_1,$$

$\Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$; ρ is the density of the fluid; g is the acceleration of gravity; E, ν, ρ_1, h_1, are the Young's modulus, the Poisson's ratio, the density and the thickness of the plate, respectively. Moreover, since the disturbance must vanish at infinity, it is required that

$$\lim_{z\to-\infty} \nabla\Phi = 0, \quad \lim_{R\to\infty} \nabla\Phi = 0 \quad (t \geq 0), \qquad (3)$$

$$R^2 = (x - \xi(t))^2 + (y - \eta(t))^2.$$

The initial conditions at $z = 0$ are:

$$\Phi = w = \partial w/\partial t = 0 \quad (t = 0). \qquad (4)$$

There are some particular cases of this problem. If the elastic parameter D is made zero, but $Q = -T$ $(T > 0)$, then the plate-covered surface reduces to the flexible membrane. If in addition also surface density of plate $M = 0$, then upper boundary of fluid becomes the free surface with surface tension. As $D = Q = 0$, the plate-covered surface reduces to the inertial surface which represents the effect of a thin uniform distribution of non-interacting floating matter, for example, broken ice. If an addition also $M = 0$, then upper boundary of fluid becomes the clean free surface.

The initial-value-problem (1)-(4) is solved by standard method. The solution of this problem can be written as

$$\Phi = \mu(t)(1/r_1 - 1/r_2) + \phi, \qquad (5)$$

where $r_1^2 = R^2 + (z - \eta(t))^2$, $r_2^2 = R^2 + (z + \eta(t))^2$. In order to obtain the formal solution for the harmonic function $\phi(\vec{x}, t)$, it is convenient to introduce a combination of the Laplace transform with respect to t and the Fourier transform with respect to spatial variables x and y.

The formal integral expression for the function $\phi(\vec{x}, t)$ can be written as

$$\phi = 2\rho \int_0^t \mu(t) \int_0^\infty \frac{\omega(k)}{\rho + Mk} e^{k(z+\eta(\tau))} J_0(kR(\tau))\times \sin(\omega(k)(t-\tau))dkd\tau, \quad (6)$$

where

$$\omega(k) = \sqrt{\frac{k(Dk^4 - Qk^2 + g\rho)}{\rho + Mk}}, \quad (7)$$

J_0 is the zeroth-order Bessel function of the first kind. If $D = Q = M = 0$, the solution (6) is consistent with the velocity potential for clean free surface and coincides with the result by Wehausen & Laitone (1960) [Eq. (13.49].

Eq. (7) is known as the dispersion relation. It is known that there is a limitation on the compressive force Q. The condition $Q < Q_* = 2\sqrt{g\rho D}$ provides steadiness of the floating elastic plate. In the present analysis, it is assumed also that $Q < Q_0 < Q_*$, where Q_0 is defined by the condition of the positive group velocity $c_g(k) = d\omega/dk$ for all wave numbers $k \geq 0$. The method of determination Q_0 was given by Bukatov (1980) for a fluid of finite depth. The value Q_0 and its attendant k_0 are found from the conditions $c_g(k_0) = dc_g(k_0)/dk = 0$. For deep water, the value k_0 is determined as the positive root of the polynomial $Dk_0^4(8Mk_0 + 15\rho) - 3g\rho^2 = 0$ and the value Q_0 is equal to

$$Q_0 = \frac{Dk_0^4(4Mk_0 + 5\rho) + g\rho^2}{k_0^2(2Mk_0 + 3\rho)}.$$

At $M = 0$, the values k_0 and Q_0 are determined explicitly

$$k_0(D/g\rho)^{1/4} = 5^{-1/4} \approx 0.669,$$

$$Q_0/\sqrt{g\rho D} = \sqrt{20}/3 \approx 1.491.$$

Fig. 1 shows the non-dimensional values $Q_0/\sqrt{g\rho D}$ and $k_0(D/g\rho)^{1/4}$. It can be seen that the values k_0 and Q_0 decrease with increasing M from zero.

All considered cases are divided into 2 groups. For elastic cover, flexible membrane and surface tension, both the phase $c_f(k) = \omega(k)/k$ and group

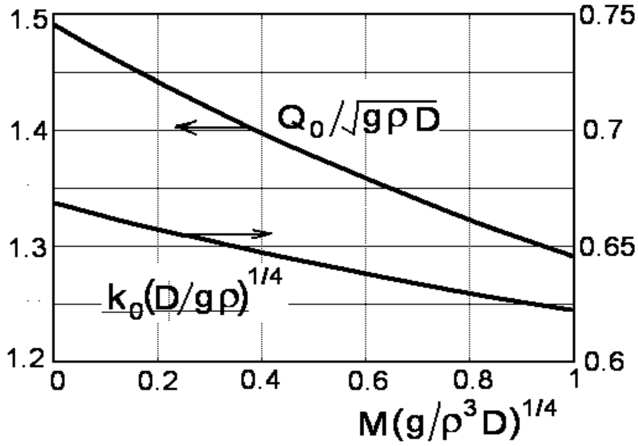

Figure 1.

velocities $c_g(k)$ have minimal values, denoted by $U_f = c_f(k_f)$ and $U_g = c_g(k_g)$, respectively. Here k_f is correspond to the wave number at which $dc_f(k_f)/dk = 0$, and analogously $k_g < k_f$ is defined by the expression $dc_g(k_g)/dk = 0$. For inertial surface and clean free surface, both the phase and group velocities are the monotone functions.

3. Velocity potential of translating and oscillating source

The velocity potential (5) can be applied for different particular cases of source motion. One of the most interesting case presents a source of oscillating strength, starting to oscillate at $t = 0$ and moving with constant velocity u in the direction Ox. The potential of a time-harmonic source with forward speed is obtained from the transient source by specifying the appropriate source strength and its motion: $\mu(t) = \mu_0 \cos\sigma t$, $\xi(t) = \xi_0 + ut$, $\eta(t) = \eta_0$, $\zeta(t) = \zeta_0$. Furthermore, a coordinate system moving with velocity u in direction Ox ($\bar{x} = x - ut$) is used. The location of the translating source is fixed in this moving system and the velocity potential (5) can be written as

$$\Phi(\bar{x}, y, z, t) = \mu_0 \cos\sigma t\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \phi(\bar{x}, y, z, t),$$

where

$$\phi = 4\int_0^{\pi/2} \int_0^t \cos\sigma(t-\tau) \int_0^\infty F(k,\theta)\times \cos(k\cos\theta(X + u\tau))\sin(\omega(k)\tau)dkd\tau d\theta, \quad (8)$$

$$F(k,\theta) = \frac{\mu_0\rho\omega(k)}{\pi(\rho + kM)} e^{k(z+\zeta_0)} \cos(kY\sin\theta),$$

$$X = \bar{x} - \xi_0,\ Y = y - \eta_0,$$

and the range of the θ-integration is reduced in the quadrant $[0, \pi/2]$. Using the function-product relations for sine and cosine, Eq. (8) becomes

$$\phi = \int_0^{\pi/2} \int_0^t \int_0^\infty F(k,\theta)\times$$

$$(\sin\Psi_1 + \sin\Psi_2 + \sin\Psi_3 + \sin\Psi_4)dkd\tau d\theta, \quad (9)$$

where

$$\Psi_{1,2}(k,\tau;t) = [\omega(k)+\sigma]\tau \pm k(X+u\tau)\cos\theta - \sigma t,$$

$$\Psi_{3,4}(k,\tau;t) = [\omega(k)-\sigma]\tau \pm k(X+u\tau)\cos\theta + \sigma t.$$

The principal physical features of the wave motion in far field can be determined by the asymptotic analysis of double integral for k and τ in (9) using the method of stationary phase. An especially important role is played by the critical (stationary) points at which

$$\partial\Psi_n/\partial k = \partial\Psi_n/\partial\tau = 0 \quad (n = 1, ..., 4).$$

The function Ψ_1 has no critical (stationary) points in the integration angle $[0, \pi/2]$. The function Ψ_2 has no more than 2 critical points. The equation

$$\omega(k) + \sigma - kU = 0 \quad (U = u\cos\theta) \qquad (10)$$

has 2 roots denoted by $k_2^{(1)}$ and $k_2^{(2)}$ only if $u > U_1(\sigma) = c_g(k_1^*)$ and $0 < \theta < \theta_1$ where the wave number k_1^* satisfies the equation $kc_g(k) - \omega(k) = \sigma$ and $\theta_1 = \cos^{-1}(U_1/u)$. It follows from the dispersion relation (7) that $k_1^* \to k_f$ and $U_1 \to U_f$ at $\sigma \to 0$. If the conditions mentioned above do not hold, the function Ψ_2 has no the critical points. The values $k_2^{(i)}$ $(i = 1, 2)$ are defined as the positive roots of the polynomial

$$Dk^5 - (Q + MU^2)k^3 - U(\rho U - 2\sigma M)k^2 +$$

$$(\rho g + 2\rho\sigma U - \sigma^2 M)k - \rho\sigma^2 = 0 \qquad (11)$$

satisfying Eq. (10). The wave motions corresponding $k_2^{(i)}$ propagate upstream $(X > 0)$ or downstream $(X < 0)$ depending on the sign of the difference $c_g(k_2^{(i)}) - U$.

The function Ψ_3 has always only one critical point. The equation

$$\omega(k) - \sigma + kU = 0 \qquad (12)$$

has one zero k_3 for any $\theta \in [0, \pi/2]$. The value k_3 is defined as the positive root of the polynomial (11) satisfying Eq. (12). The wave motion corresponding the wave number k_3 propagate always downstream.

The function Ψ_4 has no more than 3 critical points. The equation

$$\omega(k) - \sigma - kU = 0 \qquad (13)$$

has always one root $k_4^{(1)}$ and two additional roots $k_4^{(2)}$, $k_4^{(3)}$ only at $\sigma < \sigma^* = \omega(k_g) - k_g U_g$ and $U_3 < U < U_2$. The functions $U_2(\sigma)$ and $U_3(\sigma)$ are determined as follows: $U_2 = c_g(k_2^*)$, $U_3 = c_g(k_3^*)$. Here the values $k_2^* < k_g < k_3^*$ are the roots of the equation

$$\omega(k) - kc_g(k) = \sigma. \qquad (14)$$

It follows from the dispersion relation (7) that $k_2^* \to 0$, $k_3^* \to k_f$ and $U_2 \to \infty$, $U_3 \to U_f$ at $\sigma \to 0$, but k_2^*, $k_3^* \to k_g$ and U_2, $U_3 \to U_g$ at $\sigma \to \sigma^*$. If for given $\sigma < \sigma^*$ the velocity $u > U_2(\sigma)$, three roots exist for $\theta_2 < \theta < \theta_3$, however if $U_3(\sigma) < u < U_2(\sigma)$ then three roots exist only for $0 < \theta < \theta_3$, where $\theta_2 = \cos^{-1}(U_2/u)$ and $\theta_3 = \cos^{-1}(U_3/u)$. The values $k_4^{(j)}$ $(j = 1, 2, 3)$ are determined as the positive roots of the polynomial

$$Dk^5 - (Q + MU^2)k^3 - U(\rho U + 2\sigma M)k^2 +$$

$$(\rho g - 2\rho\sigma U - \sigma^2 M)k - \rho\sigma^2 = 0$$

satisfying Eq. (13). The wave motions corresponding the wave numbers $k_4^{(j)}$ propagate upstream if $c_g(k_4^{(j)}) - U > 0$ and downstream otherwise.

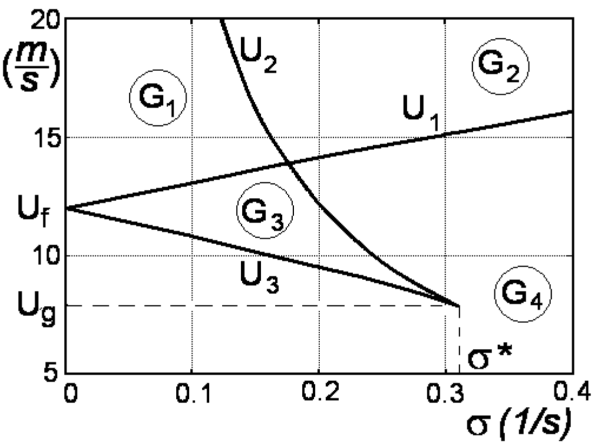

Figure 2.

Fig. 2 shows the variation of U_j $(j = 1, 2, 3)$ with σ for the case of the ice cover. The following input data are used: $E = 5\ GPa$, $\nu = 0.3$, $Q = 0$, $\rho = 1025\ kg/m^3$, $\rho_1 = 922.5\ kg/m^3$, $h_1 = 0.5\ m$. The curves U_1, U_2, U_3 divide the (σU)-plane into four regions G_n $(n = 1, ..., 4)$. There are all six waves in far field for values σ and U from the region G_1: $k_2^{(1)}, k_2^{(2)}, k_3, k_4^{(1)}, k_4^{(2)}, k_4^{(3)}$. There are four waves for the regions G_2 and G_3: $k_2^{(1)}, k_2^{(2)}, k_3$, $k_4^{(1)}$ and $k_3, k_4^{(1)}, k_4^{(2)}, k_4^{(3)}$, respectively. There are only two waves for the region G_4: $k_3, k_4^{(1)}$.

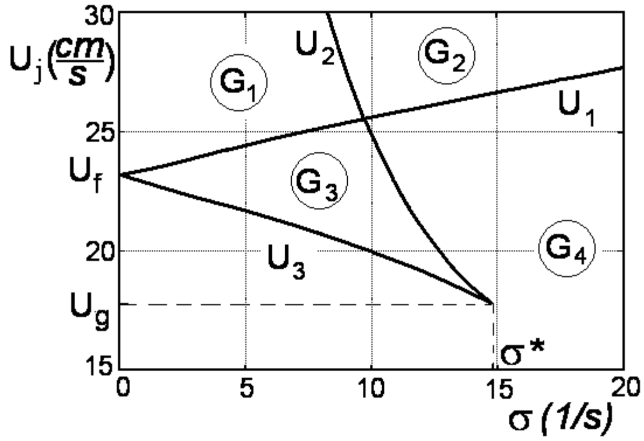

Figure 3.

Fig. 3 represents the similar picture for capillary-gravity waves. The input data for water at 20^o are used: $T = 0.0728\ N/m,\ \rho = 998\ kg/m^3$.

The basic properties of the flexural-gravity waves generated by oscillating pressure moving over ice plate were investigated by Bukatov & Cherkesov (1977), Bukatov (1980) and Bukatov & Yaroshenko (1986) for 2D and 3D problems and the fluid of finite depth. In this abstract, these results are presented in more simple form for deep water.

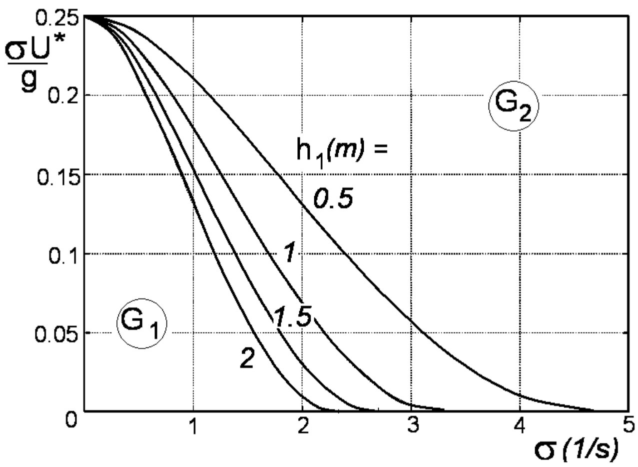

Figure 4.

For inertial surface ($D = Q = 0$), the function Ψ_2 has only one critical point, because Eq. (10) has one zero k_2 for any $\theta \in [0, \pi/2]$. The zero k_2 is always greater than k_3. For Ψ_4, there are two critical points for certain values of θ only at $\sigma < \sqrt{g\rho/M}$. Eq. (13) has 2 roots $k_4^{(1)}$ and $k_4^{(2)}$ with $k_4^{(1)} < k_4^{(2)}$ at $U < U^* = c_g(k_4^*)$ where k_4^* is the root of Eq. (14). The value k_4^* is defined as the positive root of the polynomial

$$M^3k^4 + 3\rho Mk^2(\rho + Mk) + \rho^3(k - 0.25g/U^2) = 0$$

satisfying Eq. (14). For $u < U^*$, both $k_4^{(1)}$ and $k_4^{(2)}$ exist for $\theta \in [0, \pi/2]$. However, when $u > U^*$, $k_4^{(1)}$ and $k_4^{(2)}$ exist only for $\theta > \cos^{-1}(U^*/u)$. For clean free surface ($D = Q = M = 0$), we have well known result: $k_4^* = 0.25g/U^2,\ U^* = 0.25g/\sigma$.

Numerical results for broken ice with $h_1 = 0.5, 1, 1.5, 2\ m$ are presented in Fig. 4. There are four waves for the region G_1: $k_2, k_3, k_4^{(1)}, k_4^{(2)}$. Only two waves k_2, k_3 exist for the region G_2.

The foregoing analysis is necessary, in particular, for the solution of wave radiation problem of a submerged body with forward speed. More detailed results for the hydrodynamic load of the sphere will be presented at the Workshop.

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