

# Hydrodynamic impact (Wagner) problem and Galin's theorem.

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## 1) Introduction

Galín's theorem is used to revisit the linearized Wagner problem based on the so-called flat disk approximation. Galín's theorem is mainly used for solid-solid contact when the shape of the contact surface is elliptic. We can hence reconsider the theoretical developments by Scolan and Korobkin (2001, denoted Part I in the sequel) applied to the elliptic paraboloid.

## 2) Theoretical developments

A formulation of Galín's theorem can be found in Vorovich *et al* (1974) for example, it reads:

**Theorem:** *Let the functions  $(G, F, P, I)$  defined as follows:*

$$G(x, y) = \frac{1}{R} = \frac{1}{\sqrt{x^2 + y^2}}, \quad F(x, y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad P(x, y) = \sum_{p=0}^M \sum_{q=0}^{M-p} b_{pq} x^p y^q \quad (1)$$

$$I(x, y) = \int_D \frac{P(x_0, y_0)}{\sqrt{F(x_0, y_0)}} G(x - x_0, y - y_0) dx_0 dy_0 \quad (2)$$

then, if  $(x, y) \in D = \{(x, y), \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$ ,

$$I(x, y) = \sum_{p=0}^M \sum_{q=0}^{M-p} a_{pq} x^p y^q \quad (3)$$

that means that  $I$  is a polynomial in  $(x, y)$  of the same degree as  $P$ .

A generalization of this theorem is given in Kalker (1973 and 1990). We assume in the sequel that Galín's theorem can be inverted: *If  $I$  is polynomial in  $(x, y)$ , say with degree  $M$ , then  $P$  is polynomial as well with same degree  $M$ .*

We apply this theorem in order to find the displacement potential  $\phi$  which is solution of the Boundary Value Problem formulated in Korobkin and Scolan (2006). This BVP reads

$$\begin{cases} \phi_{,xx} + \phi_{,yy} + \phi_{,zz} = 0 & z < 0 \\ \phi = 0 & z = 0, (x, y) \in \text{FS}(t) \\ \phi_{,z} = -h(t) + f(x, y) & z = 0, (x, y) \in \text{D}(t) \\ \phi \rightarrow 0 & (x^2 + y^2 + z^2) \rightarrow \infty, \end{cases} \quad (4)$$

where the regions  $\text{FS}(t)$  and  $\text{D}(t)$  are disconnected parts of the plane  $z = 0$  and correspond to the free surface and the wetted area of the body, respectively. A closed curve, which separates the regions  $\text{FS}(t)$  and  $\text{D}(t)$ , is denoted  $\Gamma(t)$  and is referred to as the contact line. The Neumann condition on  $\text{D}(t)$  depends on the penetration depth  $h$  and on the shape function  $z = f(x, y)$  of the penetrating body. The origin of the coordinate system in which the shape is defined is centered at the initial contact point. From the BVP (4) we derive an integral equation

$$\frac{1}{2\pi} \int_{\text{D}(t)} \frac{S(x_0, y_0, t) dx_0 dy_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = h(t) - f(x, y) \quad (x, y) \in \text{D}(t), \quad (5)$$

where  $S(x_0, y_0, t)$  denotes the planar Laplacian of  $\phi$ :  $S(x, y) = \phi_{,xx} + \phi_{,yy}$ . By solving the integral equation (5), we obtain a Poisson equation for  $\phi$  which satisfies the following boundary condition

$$\phi = 0 \quad \text{on } \Gamma(t). \quad (6)$$

Additional condition is suggested to prescribe along the contact line. This condition implies that not only the displacement potential  $\phi(x, y, 0, t)$  and the vertical displacement  $\phi_{,z}(x, y, 0, t)$  but also the horizontal

displacements  $\phi_{,x}(x, y, 0, t)$  and  $\phi_{,y}(x, y, 0, t)$  are continuous through  $\Gamma(t)$ . With account for (6) the latter condition can be presented as

$$\phi_{,n} = 0 \quad \text{on } \Gamma(t) \quad (7)$$

where  $\phi_{,n} = \phi_{,x}(x, y, 0, t)n_x + \phi_{,y}(x, y, 0, t)n_y$  and  $\mathbf{n} = (n_x, n_y)$  is the unit normal vector along the contact line  $\Gamma(t)$ .

### 3) Application of Galin's theorem

We now consider the elliptic paraboloid whose shape function is given by

$$f(x, y) = \frac{x^2}{A^2} + \frac{y^2}{B^2} \quad (8)$$

where  $(A^2, B^2)$  are twice the curvature radii at the initial contact point, denoted  $(r_x, r_y)$ . The RHS of (5) is a polynomial of degree 2. Then, according to Galin's theorem, the planar Laplacian  $S$  can be written as a polynomial of degree 2 as well, say

$$S(x, y) = \frac{1}{\sqrt{F}} (b_{00} + b_{20}x^2 + b_{02}y^2) \quad (9)$$

where  $b_{00}$ ,  $b_{20}$  and  $b_{02}$  are unknown coefficients at that stage. Vorovich *et al* (1974) detailed the calculation of the polynomials  $I(x, y)$  in equation (2). The identification of each coefficient of 1,  $x^2$  and  $y^2$  on both hands of equation (5) leads to

$$2b_{00}bS_{00} + b_{20}b^3S_{01} + b_{02}b^3S_{10} = 2h \quad (10)$$

$$b_{20} (2bS_{20} - k^2bS_{11}) + b_{02} (2k^4bS_{11} - k^2bS_{20}) = -\frac{2}{A^2} \quad (11)$$

$$b_{20} (2bS_{11} - k^2bS_{02}) + b_{02} (2k^4bS_{02} - k^2bS_{11}) = -\frac{2}{B^2} \quad (12)$$

where  $b = ka$ ,  $k^2 = 1 - e^2$  and  $S_{ij}$  are known functions of eccentricity  $e$  in terms of standard Elliptic Integral functions  $K(e)$  and  $E(e)$ . The three equations (10) to (12) are used to calculate  $(b_{00}, b_{20}, b_{02})$  in terms of  $(A, B, h, a, b)$ . However it should be noted that  $(a, b)$  are still unknown. The expression of the function  $S$  suggests that the displacement potential itself reads

$$\phi(x, y, t) = H(x, y, t) \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{3/2} \quad (13)$$

where function  $H$  is some unknown function. Some elements of proof of that result can be found in Sclan and Korobkin (2008). The boundary conditions (6) and (7) on  $\Gamma(t)$  are hence implicitly accounted for. By taking the spatial derivatives of  $\phi$ , and arriving at the planar Laplacian, we conclude that  $H$  is a constant only function of time  $t$ , thus  $H$  does not depend on spatial coordinates  $(x, y)$ .

$$\Delta_2\phi(x, y, t) = \frac{3H}{\sqrt{F}} \left( -\frac{1}{a^2} - \frac{1}{b^2} + \frac{x^2}{a^2} \left( \frac{2}{a^2} + \frac{1}{b^2} \right) + \frac{y^2}{b^2} \left( \frac{1}{a^2} + \frac{2}{b^2} \right) \right) \quad (14)$$

Identifying (14) and (9) lead to 3 additional equations to already established equations (10) to (12).

$$b_{00} = -3H \left( \frac{1}{a^2} + \frac{1}{b^2} \right), \quad b_{20} = \frac{3H}{a^2} \left( \frac{2}{a^2} + \frac{1}{b^2} \right), \quad b_{02} = \frac{3H}{b^2} \left( \frac{1}{a^2} + \frac{2}{b^2} \right) \quad (15)$$

That is enough to obtain  $(b_{00}, b_{20}, b_{02})$ ,  $(a, b)$  and  $H$  in terms of  $(A, B, h)$ . In particular we can prove that  $H$  is proportional to  $h$ , hence the displacement potential varies linearly with the penetration depth. The final equations to be solved are

$$H = -\frac{2hb}{3E} \quad (16)$$

$$\frac{f_A(e)}{b^2} = \frac{1}{hA^2}, \quad f_A(e) = \frac{1 - e^2}{e^2E} ((2e^2 - 1)E(e) + (1 - e^2)K(e)) = k^2 \left( 1 + k^2 \frac{D}{E} \right) \quad (17)$$

$$\frac{f_B(e)}{b^2} = \frac{1}{hB^2}, \quad f_B(e) = \frac{1}{e^2E} ((1 + e^2)E(e) + (e^2 - 1)K(e)) = 2 - k^2 \frac{D}{E} \quad (18)$$

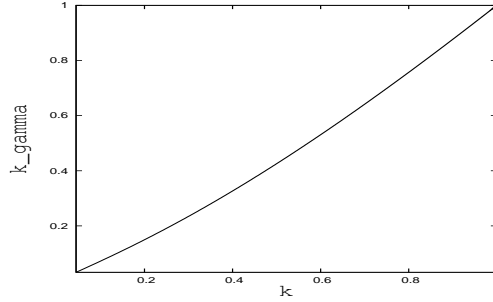
where  $D = (K - E)/e^2$ . From these last equations, we get

$$\frac{B^2}{A^2} = k^2 \frac{1 + k^2 D/E}{2 - k^2 D/E} = \frac{f_A}{f_B} \quad (19)$$

which is exactly equations (6.13) or (6.22) in Part I. The displacement potential is

$$\phi(x, y, t) = -\frac{2hb}{3E} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2} \quad (20)$$

It is clear that no special assumptions have been made regarding the kinematics of the entering body. We hence conclude that elliptic paraboloid entering liquid with arbitrary kinematics provide elliptic contact line, always. Equation (19) proves that given the aspect ratio  $k_\gamma = B/A$ , we obtain a unique aspect ratio  $k = b/a$  as illustrated in the figure below



As a consequence neither  $k$ , nor  $e$  depend on time, yielding

$$2h\dot{b} = \dot{h}b \quad \Rightarrow \quad h = \alpha b^2 \quad (21)$$

and the coefficient  $\alpha$  is obtained from one of the equations (17) or (18), yielding

$$b(t) = \sqrt{h}A\sqrt{1 + k^2 D/E} \quad (22)$$

similar to eq. (6.14) in Part I. Without any assumptions, it remains an uncertainty regarding the expression of the velocity potential (noted here  $\varphi$ ) which should be given by eq. (4.10) of Part I. The time derivative of the displacement potential reads

$$\varphi = \frac{d\phi}{dt} = \sqrt{F} \left( \dot{H}F + \frac{3}{2}H\dot{F} \right) \quad (23)$$

where  $F$  is defined by equation (1). The coefficient in brackets should be independent on spatial coordinates  $(x, y)$ . Therefore, by using (16) and  $\dot{b}$  is given by (21)

$$\dot{H} = -\frac{2h\dot{b}}{E} = -\frac{\dot{h}b}{E} - \frac{hb\dot{f}_A}{E f_A} \quad (24)$$

which means that we necessarily impose that  $\dot{f}_A \equiv 0$  and  $\dot{f}_B \equiv 0$  at any time. In other words an elliptic paraboloid provides elliptic contact lines.

#### 4) 3DoF motion of elliptic paraboloid

In this section, the elliptic paraboloid enters liquid with a vertical velocity  $\dot{h}(t)$ , a horizontal velocity  $U = \dot{x}_o$  and it starts a rotation around the  $y$ -direction when it hits the liquid. The inclination angle is  $\alpha(t)$  with  $\alpha(0) = 0$ .

$$z = -h(t) + X \sin \alpha + Z \cos \alpha, \quad x = x_o(t) + X \cos \alpha - Z \sin \alpha, \quad y = Y \quad (25)$$

where  $X, Y, Z$  are the local and  $x, y, z$  global coordinates. We assume that  $h(t)/R_x \ll 1$ ,  $\alpha = O(\sqrt{h/R_x})$ ,  $X, Y, x_o(t) = O(\sqrt{hR_x})$ , and  $Z = O(h)$  in the contact region between that entering body and the liquid.

Then  $X \sin \alpha = O(h)$ ,  $\cos \alpha = 1 + O(h/R_x)$ ,  $Z \sin \alpha / (X \cos \alpha) = O(h/R_x)$ . Neglecting terms in (25) with relative order of  $O(h/R_x)$  and higher, we obtain

$$x = x_o(t) + X, \quad z = -h(t) + X\alpha(t) + X^2/A^2 + Y^2/B^2, \quad (26)$$

and finally in the leading order

$$z = - \left( h(t) + \frac{1}{2} R_x \alpha^2(t) \right) + \frac{(x - x_o(t) + \alpha R_x)^2}{2R_x} + \frac{y^2}{2R_y} \quad (27)$$

The latter equation describes an elliptic paraboloid with the same curvature radius than the initial one described by equation (8). It is convenient to introduce a modified penetration depth  $\tilde{h}(t)$  and the new global coordinates  $\tilde{x}, \tilde{y}$  as follows

$$\tilde{h}(t) = h + \frac{1}{2} R_x \alpha^2, \quad x = x_o - \alpha R_x + \tilde{x} \sqrt{\tilde{h} R_y}, \quad y = \tilde{y} \sqrt{\tilde{h} R_y}, \quad z = \tilde{z} \sqrt{\tilde{h} R_y}, \quad (28)$$

yielding a new displacement potential  $\tilde{\phi}$  (as described in Korobkin, 2002) which verifies a new Neumann condition on the corrected wetted surface  $\tilde{D} = \left\{ (x - x_o(t) + \alpha R_x)^2 k^2 + y^2 < 2\tilde{h}(t) R_x (1 + k^2 D/E) \right\}$

$$\phi(x, y, z, t) = \tilde{h} \sqrt{\tilde{h} r_y} \tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{z}), \quad \tilde{\phi}_{,\tilde{z}} = -1 + \frac{1}{2} \left( \frac{R_y}{R_x} \right) \tilde{x}^2 + \frac{1}{2} \tilde{y}^2 \quad (\tilde{z} = 0, (\tilde{x}, \tilde{y}) \in \tilde{D}). \quad (29)$$

The problem of water entry with 3DoF has been reduced to the problem of vertical entry with constant speed. The potential  $\tilde{\phi}$  depends on the only parameter  $r_y/r_x$  and is given by (20). However, the velocity potential and the corresponding pressure distribution in the contact region are far from being trivial due to the body rotation and its translation in  $x$ -direction. The impact stage with jetting all along the periphery of the contact region lasts until the normal velocity of the contact line is zero at a single point, which is expected to be the rear point of the contact line if  $\dot{x}_0(t) > 0$ . The pressure in the contact region is of particular interest with the zone of negative pressures starting at a point inside the region and approaching the contact line at the rear point. A motion of an elliptic paraboloid with six degrees of freedom is considered in a similar way without difficulties.

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## 5) Conclusion

Galin's theorem appears as a powerful shortcut to prove some results exposed in Part I, at least when the shape function can be represented as even polynomials of cartesian coordinates. However this theorem is of no use as soon as the shape is pointed as for a cone. As a consequence of Galin's theorem, it is also shown that the entry of an elliptic paraboloid, animated along its 3DoF and even 6DoF, leads to elliptic contact line and its eccentricity does not depend on the motions. Another question arises whether or not Galin's theorem can be extended to more complicated shape than an ellipse, for example a shape whose contour is not convex. Preliminary calculations let us think that the problem is very tricky...

## 6) References

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