The Time-Dependent Motion of a Floating Cylinder

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1 Introduction

In two papers (Ursell, 1964; Maskell & Ursell, 1970) a solution method in the time domain was presented for the case of a two-dimensional floating cylinder half submerged based on the frequency domain solutions using the Fourier/Laplace transform. An approximation of the solution based on a *singularity expansion method* was also given, accurate for medium time scales. The present work may be thought of as a follow up to Ursell (1964); Maskell & Ursell (1970). In particular we investigate in detail the connection between the solution method of (Ursell, 1964; Maskell & Ursell, 1970), the method recently developed based on the generalized eigenfunction expansion (Fitzgerald & Meylan, 2011), and *Cummins method* (Cummins, 1962; Ogilvie, 1964). We also investigate two formulations of the singularity expansion method.

2 A floating half-immersed cylinder

We consider a circular cylinder of radius a whose equilibrium position of its centre lies on the mean free surface. The fluid depth is assumed infinite, and the cylinder is constrained to move only in heave. We assume that we can describe the motion of the fluid by a velocity potential $\Phi(x, z; t)$. In what follows we will use two coordinate systems, a Cartesian (x, z) and a polar (r, θ) both centered at the center of the cylinder. The displacement, measured in the z direction is given by $Z(t)$ and the surface displacement (also measured in the z direction) is given by $H(x, t)$. The equations of motion are

$$\Delta\Phi(x, z; t) = 0, \quad z < 0, r > a, \tag{1a}$$

$$|\partial_z\Phi| \to 0, \quad z \to -\infty, \tag{1b}$$

subject to the linearized Bernoulli condition,

$$\partial_t\Phi + gH = 0, \quad z = 0, r > a,$$

where g is the gravitational acceleration, and the linearized kinematic condition,

$$\partial_z\Phi = \partial_t H, \quad z = 0, r > a. \tag{1c}$$

On the cylinder the radial velocity components of the body and of the fluid are equal,

$$-\partial_r\Phi = \partial_t Z(t)\cos\theta, \quad r = a, -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{1d}$$

The equation of motion of the body is

$$M\partial_t^2 Z(t) = -CZ(t) + \rho a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \partial_t\Phi(a\sin\theta, -a\cos\theta; t)\cos\theta\,\mathrm{d}\theta, \tag{2}$$

where $M = \frac{1}{2}\pi\rho a^2$ and $C = 2\rho g a$ are the mass and hydrostatic restoring force respectively (ρ is the fluid density).

3 Frequency domain solution

We introduce the following notation

$$\Phi(x, z; t) = \mathrm{Re}\left[\phi(x, z; \omega)\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{3a}$$

$$H(x, t) = \mathrm{Re}\left[\eta(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{3b}$$

$$Z(t) = \mathrm{Re}\left[\zeta(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{3c}$$

which transforms equations (1a) to (2) to

$$\Delta\phi(x, z; \omega) = 0, \quad z < 0, r > a, \tag{4a}$$

$$|\partial_z\phi| \to 0, \quad z \to -\infty, \tag{4b}$$

$$\partial_z\phi = k\phi, \quad z = 0, r > a, \tag{4c}$$

where $k = \omega^2/g$ and

$$-M\omega^2\zeta(\omega) = -C\zeta(\omega) - \mathrm{i}\rho a\omega \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(a\sin\theta, -a\cos\theta; \omega)\cos\theta\,\mathrm{d}\theta. \tag{5}$$

These equations are subject to radiation conditions as $|x| \to \infty$. We assume that we have symmetric or anti-symmetric incident waves (as opposed to waves incident from the left or right). Since the body is symmetric, the anti-symmetric incident wave will not excite any body motion. The symmetric incident wave is given by

$$\phi_{\mathrm{s}}^{\mathrm{I}} = \frac{\mathrm{i}g}{\omega}\cos(kx)\mathrm{e}^{kz}, \tag{6}$$

and the anti-symmetric incident wave is given by

$$\phi_{\mathrm{a}}^{\mathrm{I}} = \frac{\mathrm{i}g}{\omega}\sin(kx)\mathrm{e}^{kz}. \tag{7}$$

Corresponding to each incident wave is a diffracted potential given by the solution to

$$\Delta\phi_{\mathrm{a,s}}^{\mathrm{D}}(x,z;\omega) = 0, \quad z < 0, r > a, \tag{8a}$$

$$\partial_z\phi_{\mathrm{a,s}}^{\mathrm{D}} \to 0, \quad z \to -\infty, \tag{8b}$$

$$\partial_z\phi_{\mathrm{a,s}}^{\mathrm{D}} = k\phi_{\mathrm{a,s}}^{\mathrm{D}}, \;\; z = 0, \;\; |x| > a, \tag{8c}$$

$$\partial_r\phi_{\mathrm{a,s}}^{\mathrm{D}} = -\partial_r\phi_{\mathrm{a,s}}^{\mathrm{I}} \quad r = a, -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{8d}$$

We now define the radiation potential ϕ^{R} as the solution to

$$\Delta\phi^{\mathrm{R}}(x,z;\omega) = 0, \quad z < 0, r > a, \tag{9a}$$

$$\partial_z\phi^{\mathrm{R}} \to 0, \quad z \to -\infty, \tag{9b}$$

$$\partial_z\phi^{\mathrm{R}} = k\phi^{\mathrm{R}}, \quad z = 0, r > a, \tag{9c}$$

$$\partial_r\phi^{\mathrm{R}} = \cos\theta \quad r = a, -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{9d}$$

Both the diffraction and radiation potentials are subject to the Sommerfeld radiation condition.

The solution for the anti-symmetric velocity potential is

$$\phi_{\mathrm{a}} = \phi_{\mathrm{a}}^{\mathrm{I}} + \phi_{\mathrm{a}}^{\mathrm{D}}. \tag{10}$$

The solution for the symmetric potential is

$$\phi_{\mathrm{s}} = \phi_{\mathrm{s}}^{\mathrm{I}} + \phi_{\mathrm{s}}^{\mathrm{D}} - \mathrm{i}\omega\zeta_{\mathrm{s}}(\omega)\phi^{\mathrm{R}}. \tag{11}$$

We define $A(\omega)$ (the added mass) and $B(\omega)$ (the damping) by

$$A(\omega) + \frac{\mathrm{i}}{\omega}B(\omega) = \rho a\int_{-\pi/2}^{\pi/2}\phi^{\mathrm{R}}\cos\theta\,\mathrm{d}\theta. \tag{12}$$

The solution for $\zeta_{\mathrm{s}}(\omega)$ is given, writing the equation in standard notation for floating bodies

$$\left(-\omega^2 M + C - \omega^2 A(\omega) - \mathrm{i}\omega B(\omega)\right)\zeta_{\mathrm{s}}(\omega) = f_{\mathrm{s}}, \tag{13}$$

where

$$f_{\mathrm{s}} = -\mathrm{i}\omega\rho a\int_{-\pi/2}^{\pi/2}\left(\phi_{\mathrm{s}}^{\mathrm{I}} + \phi_{\mathrm{s}}^{\mathrm{D}}\right)\cos\theta d\theta. \tag{14}$$

4 Time-domain Solution

At $t = 0$ we specify that

$$Z|_{t=0} = Z_0, \quad \partial_t Z|_{t=0} = \dot{Z}_0. \tag{15}$$

4.1 Solution by Generalized Eigenfunction Expansion

The solution for a floating body by the generalized eigenfunction method has been recently developed (Fitzgerald & Meylan, 2011). We denote the frequency-domain body displacement solution for the symmetric incident wave by ζ_{s}. The time-dependent displacement is

$$Z(t) = \frac{2C}{\pi\rho g^2}\int_{\mathbb{R}^+}\omega\left[Z_0\cos(\omega t) + \dot{Z}_0\frac{\sin(\omega t)}{\omega}\right]|\zeta_{\mathrm{s}}(\omega)|^2\,d\omega. \tag{16}$$

Equation (16) allows us to find the long time asymptotics. We know that as $\omega \to 0$, $\eta_s(\omega) \to 1$ so that if we apply integration by parts twice (assuming $\dot{Z}_0 = 0$) we obtain

$$Z(t) = \frac{2CZ_0}{\pi\rho g^2}\left(-\frac{1}{t^2} - \int_{\mathbb{R}^+}\partial_\omega^2\left(\omega|\eta_{\mathrm{s}}(\omega)|^2\right)\frac{\cos(\omega t)}{t^2}\,\mathrm{d}\omega\right), \tag{17}$$

which gives exactly the expression for the long time asymptotics as obtained by Ursell (1964). This derivation also complements the derivation of the asymptotics given by McIver & McIver (2010).

4.2 Solution by Fourier/Laplace transform

We derive here the solution by a Fourier/Laplace transform as was done by Ursell (1964); Maskell & Ursell (1970). We define the Fourier/Laplace transform $\hat{Z}(s)$ as

$$\hat{Z}(s) = \int_0^\infty Z(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t, \tag{18}$$

If we substitute this transformation into equations (1a) to (2) we obtain

$$\hat{Z}(s) = \frac{(\mathrm{i}sz_0 - \dot{z}_0)\left(-M - A(s) - \frac{\mathrm{i}}{s}B(s)\right)}{-s^2M + C - s^2A(s) - \mathrm{i}sB(s)}. \tag{19}$$

The expression for the initial displacement is identical to that obtained by Ursell (1964); Maskell & Ursell (1970).

We introduce the notation

$$\hat{Y}(s) = \frac{\mathrm{i}s\left(-M - A(s) - \frac{\mathrm{i}}{s}B(s)\right)}{-s^2M + C - s^2A(s) - \mathrm{i}sB(s)}. \quad (20)$$

From the property that $\hat{Y}(s) = \hat{Y}(-s)^\star$ we can write the inverse transformation as

$$Z(t) = Z_0\Big(\frac{1}{\pi}\int_{\mathbb{R}^+} \mathrm{Re}\,(\hat{Y}(s))\cos st\,\mathrm{d}s + \frac{1}{\pi}\int_{\mathbb{R}^+} \mathrm{Im}\,(\hat{Y}(s))\sin st\,\mathrm{d}s\Big). \quad (21)$$

From the Kramers-Kronig relations (Mei, 1989) we know that

$$\frac{1}{\pi}\int_{\mathbb{R}^+} \mathrm{Re}\,(\hat{Y}(s))\cos st\,\mathrm{d}s = \frac{1}{\pi}\int_{\mathbb{R}^+} \mathrm{Im}\,(\hat{Y}(s))\sin st\,\mathrm{d}s. \quad (22)$$

We can show further that

$$\mathrm{Re}\,(\hat{Y}(s)) = \frac{CB}{|-s^2M + C - s^2A - \mathrm{i}sB|^2} \quad (23)$$
$$= \frac{2C\omega}{\rho g^2}|\zeta_{\mathrm{s}}(\omega)|^2 \quad (24)$$

the last line following from the *Haskind-Hanoka* relations (Mei, 1989) that

$$B = \frac{\omega|f_{\mathrm{s}}|^2}{\rho g^2}. \quad (25)$$

This of course implies that the solution by the generalized eigenfunction method and the Fourier/Laplace transform are the same. We also note that the cosine integral in equation (21) is much easier to compute numerically than the sine integral.

4.3 Connection with the Cummins formulation

We can write the inverse Laplace transform as an integral equation as follows. We write equation (19) as

$$(M + A(\infty))\left(-s^2\hat{Z} + \mathrm{i}sz_0 - \dot{z}_0\right) = -C\hat{Z} - \left(-s^2\hat{Z} + \mathrm{i}sz_0 - \dot{z}_0\right)\left(A(s) - A(\infty) + \frac{\mathrm{i}}{s}B(s)\right), \quad (26)$$

where the reason for the introduction of the added mass at infinity will become apparent shortly. We now introduce the following function

$$L(t) = \frac{2}{\pi}\int_0^\infty \frac{B(\omega)}{\omega}\sin(\omega t)\,\mathrm{d}\omega, \quad (27)$$

whose Fourier transform is given by

$$\hat{L} = A(s) - A(\infty) + \frac{\mathrm{i}}{s}B(s), \quad (28)$$

(Mei, 1989). This gives us

$$(M + A(\infty))\left(-s^2\hat{Z} - \mathrm{i}sz_0 - \dot{z}_0\right) = -C\hat{Z} - \hat{L}\left(-s^2\hat{Z} + \mathrm{i}sz_0 - \dot{z}_0\right), \quad (29)$$

which, taking the inverse Fourier transform, gives us

$$(M + A(\infty))\,\partial_t^2 Z + \int_0^t \partial_\tau^2 Z L(t-\tau)\,\mathrm{d}\tau + CZ = 0. \quad (30)$$

This is exactly the equation derived by Cummins. However, the derivation presented here is quite different and in many ways more straightforward than the standard derivation (Mei, 1989). This derivation method has appeared previously in Meylan & Sturova (2009); McIver & McIver (2010).

5 Approximation of the Solution using Complex Poles

In Maskell & Ursell (1970) an asymptotic expression for the solution for medium times was given by a contour deformation and by calculating the residue at a single pole. This method is known as the *singularity expansion method*. A second method to derive an approximate formula for the solution based on the generalized eigenfunction expansion was presented in Meylan & Eatock Taylor (2009). We can approximate the solution for displacement in the frequency domain near this pole by

$$\zeta(\omega) \approx \frac{\bar{\zeta}}{\omega - \omega_0}, \quad (31)$$

or we can approximate the Fourier/Laplace solution near this pole by

$$\hat{Y}(s) \approx \frac{\bar{Y}}{s - \omega_0}, \quad (32)$$

where $\bar{\zeta}$ or $\bar{Y}$ are the residues (which we calculate numerically). If we substitute these approximations into equations (16) or the Fourier inversion of $Y(s)$ and consider only the contributions from the poles we obtain

$$Z(t) \approx \mathrm{Re}\left\{\frac{\omega_0}{\omega_0 - \omega_0^\star}|\bar{\zeta}|^2 e^{-\mathrm{i}\omega_0 t}\right\}, \quad (33)$$

or

$$Z(t) \approx \text{Re}\left\{\bar{Y} e^{-\mathrm{i}\omega_0 t}\right\}. \quad (34)$$

Figure 1 shows the results using these two approximations. We can see that the second one performs much better. This implies that the singularity expansion or equivalent methods are best derived from the Fourier/Laplace solution and furthermore that both the cosine and sine terms should be retained when making this approximation.

We can also consider the case of an incident wave packet. In this case the generalized eigenfunction expansion simplifies and the solution in the time domain can be written as

$$Z(t) = \text{Re}\left\{\frac{1}{\pi}\int_{\mathbb{R}^+} \hat{f}(\omega)\zeta(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\right\}, \quad (35)$$

where $\hat{f}(\omega)$ is the Fourier transform of the undisturbed incident wave packet at $t = 0$. We consider here only symmetric incident wave packets, but note that any packet incident from a single direction can be decomposed into a symmetric and antisymmetric wave packet and there is no excitation by the anti-symmetric packet. We can substitute the approximation (31) into this equation and we obtain

$$Z(t) \approx \text{Re}\left\{2\mathrm{i}\hat{f}(\omega_0)\bar{\zeta}\right\}. \quad (36)$$

Figure 2 shows the solution for this case for

$$\hat{f}(\omega) = \exp(-10(\omega - \text{Re}\,\{\omega_0\})^2). \quad (37)$$

We can see that, in this case, the approximation works well.

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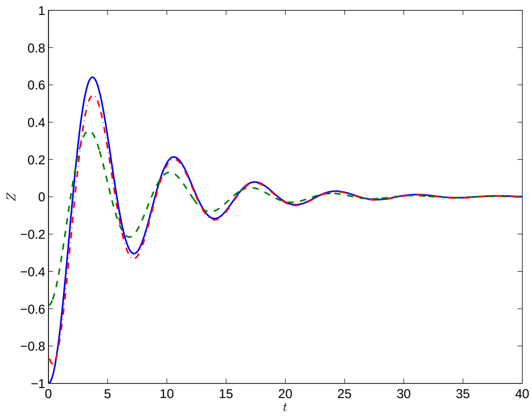

Figure 1: The displacement Z as a function of t (solid line), the approximation derived from the generalized eigenfunction expansion (dashed line) and the approximation derived from the Fourier/Laplace transform solution (chained line).

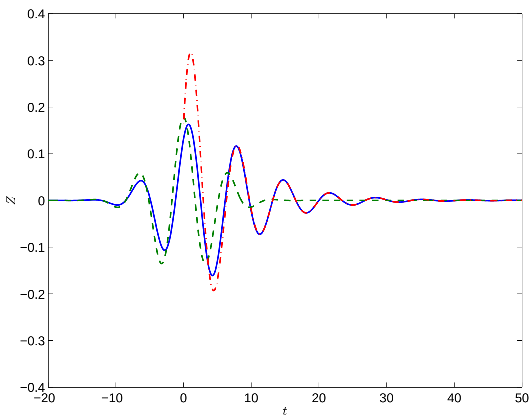

Figure 2: The displacement Z as a function of t for an incident wave packet (solid line). Also shown is the wave packet elevation at $x = 0$ if the body was absent (dashed line) and an approximate solution based on an expansion of $\zeta(\omega)$ (chained line).