Hydrodynamic exciting forces on immersed prolate spheroids

by

Spyros A. Mavrakos
mavrakos@naval.ntua.gr

Ioannis K. Chatjigeorgiou
chatzi@naval.ntua.gr

Laboratory for Floating Bodies and Mooring Systems,
School of Naval Architecture and Marine Engineering, National Technical University of Athens,
9 Heroon Polytechniou Ave, Zografos Campus, 157-73, Athens, Greece

1 Introduction

It is the purpose of this study to provide the analytical solution for the hydrodynamic diffraction problem on submerged prolate spheroids in infinite water depth. The final goal is to calculate analytically the hydrodynamic exciting forces acting on the spheroid in regular waves. Briefly, the solution method is based on the multipole expansions and employs the multipole potentials derived by Thorne [1]. Thorne's formulas describe the velocity potential at singular points within a fluid domain with free upper surface. The singularities at these points are characterized by their giving rise to potentials which are typical singular solutions of Laplace's equation in the neighborhood of the singularity. In addition, multipole potentials satisfy the free surface and bottom boundary conditions and behave like outgoing waves from the singular point which in the case of spheres or spheroids will be the center of the body.

The main challenge in the analytical process is the requirement to express the multipole potentials with respect to the coordinates of the investigated geometry. That was indeed proven a difficult task as Thorne's formulas engage terms expressed with respect to both spherical and polar coordinates. The employment of Thorne's multipole potentials does not pose severe difficulties as far as spheres are concerned as only the polar terms need to be manipulated and that has been effectively realized several times in the past (see e.g. [2-5]). For spheroidal bodies however, the multipole potentials need to be transformed into the associated spheroidal coordinates and inevitably this task requires the derivation of the appropriate addition theorems.

2 Potentials expressed as multipole expansions

The prolate spheroid (Fig. 1) is located at a distance f below the undisturbed free surface. A Cartesian coordinate system (x, y, z) fixed on the free surface is defined with its vertical z axis pointing downwards (Fig. 2). The associated Cartesian coordinate system fixed on the center of the spheroid is denoted by (x, y, z^*) so that $z=z^*+f$.

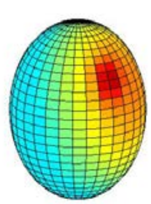

Fig. 1. 3D image of a prolate spheroid

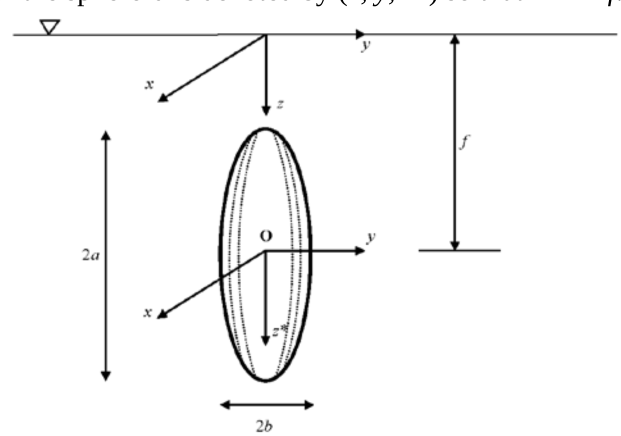

Fig. 2. Sketch of the spheroid; coordinate systems and geometrical definitions

The analysis of the hydrodynamic problem in infinite water depth starts with the diffraction component which is expressed as a multipole expansion according to $\phi_D(u,\vartheta,\psi)=\omega A e^{-Kf}\sum_{m=0}^{\infty}\hat{\phi}_D^m(u,\vartheta)\cos m\psi$ where $0\leq u<\infty$, $0\leq\vartheta\leq\pi$, $0\leq\psi<2\pi$ are the prolate spheroidal coordinates, ω is the wave frequency, A is the linear amplitude of the incident waves, f is the immersion depth and $K=\omega^2/g$. Wang [2] used the method of Havelock [6] and wrote the generalized velocity potential $\hat{\phi}_D^m$ as $\hat{\phi}_D^m=a^{m+2}F_m^m\hat{\phi}_m^m+\sum_{n=m+1}^{\infty}a^{n+2}F_n^m\hat{\Omega}_n^m$ where F_n^m are unknown expansion coefficients. a herein is the semi-major axis of the spheroid whereas the multipole potentials $\hat{\phi}_m^m$ and $\hat{\Omega}_n^m$ are given by

$$\hat{\phi}_m^m=\hat{\psi}_m^m+i\hat{\chi}_m^m=\frac{P_m^m(\cos\theta)}{r^{m+1}}+\frac{P_m^m(\cos\theta')}{r'^{m+1}}+2K\times PV\int_0^{\infty}\frac{k^m}{k-K}e^{-k(z+f)}\mathrm{J}_m(kR)\mathrm{d}k+2\pi iK^{m+1}e^{-K(z+f)}\mathrm{J}_m(KR) \quad (1)$$

$$\hat{\Omega}_n^m=\frac{P_n^m(\cos\theta)}{r^{n+1}}+\frac{K}{n-m}\frac{P_{n-1}^m(\cos\theta)}{r^n}+(-1)^{m+n}\left[\frac{P_n^m(\cos\theta')}{r'^{n+1}}+\frac{K}{n-m}\frac{P_{n-1}^m(\cos\theta')}{r'^n}\right] \quad (2)$$

In Eqs. (1)-(2) (r,θ) are spherical coordinates fixed at the center of the body whereas (r',θ') are also spherical coordinates that refer to image point $(0,0,-f)$. Also $R=c\sinh u\sin\vartheta$ is the polar radius, J_m is the mth order Bessel function of the first kind, P_n^m is the associated Legendre function of the first kind with order m and degree n and finally PV denotes that Cauchy's principal value of the integral is taken. Apparently, in order to apply the zero velocity condition on body's surface $\partial\phi_D/\partial u=-\partial\phi_I/\partial u$, the multipole potentials given by Eqs. (1) and (2) must be transformed into prolate spheroidal coordinates. It is noted that ϕ_I denotes the incident wave potential while the boundary condition on the wall must be applied at $u=u_0$ with $u_0=\mathrm{atanh}(b/a)$ where b is the semi-minor axis of the spheroid. The incident wave potential is also expressed as a multipole expansion $\phi_I(u,\vartheta,\psi)=\omega A e^{-Kf}\sum_{m=0}^{\infty}\phi_I^m(u,\vartheta)\cos m\psi$, with $\phi_I^m=(1/K)\varepsilon_m i^m e^{-Kz*}\mathrm{J}_m(KR)$, where $\varepsilon_m=1$ for $m=0$ and $\varepsilon_m=2$ for $m\geq1$. To achieve proper transformation of the multipole potentials into prolate spheroidal coordinates, the following theorems were proven using Cook's work [7-8]

$$\frac{P_n^m(\cos\theta)}{r^{n+1}}=\frac{(2/c)^{n+1}}{(n-m)!\pi^{1/2}}\sum_{s=m}^{\infty}(-1)^s\frac{(1/2+n+2s-2m)\Gamma(1/2+n+s-m)\Gamma(n+2s-3m+1)}{\Gamma(s-m+1)\Gamma(n+2s-m+1)}$$
$$\times P_{n+2s-2m}^m(\mu)Q_{n+2s-2m}^m(\xi) \quad (3)$$

$$\hat{\psi}_m^m=\frac{(2/c)^{m+1}}{\pi^{1/2}}\sum_{s=m}^{\infty}(-1)^s\frac{(1/2+2s-m)\Gamma(s+1/2)\Gamma(2s-2m+1)}{\Gamma(s-m+1)\Gamma(2s+1)}P_{2s-m}^m(\mu)Q_{2s-m}^m(\xi)$$
$$+\sqrt{\frac{\pi}{2c}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}J(Kf;m,s)P_s^m(\mu)P_s^m(\xi) \quad (4)$$

$$\hat{\chi}_m^m=2\pi K^{m+1}e^{-2Kf}\sqrt{\frac{\pi}{2Kc}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}\mathrm{I}_{s+1/2}(Kc)P_s^m(\mu)P_s^m(\xi) \quad (5)$$

$$(-1)^{m+n}\left[\frac{P_n^m(\cos\theta')}{r'^{n+1}}+\frac{K}{n-m}\frac{P_{n-1}^m(\cos\theta')}{r'^n}\right]=\frac{1}{(n-m)!}\sqrt{\frac{\pi}{2c}}\sum_{s=m}^{\infty}(-1)^{n+s}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}$$
$$\times(A_{ns}+KA_{n-1,s})P_s^m(\mu)P_s^m(\xi) \quad (6)$$

In Eq. (3) Q_n^m is the associated Legendre function of the second kind of order m and degree n. It can be shown that the coefficients $J(Kf;m,s)$ and $A_{n,s}$ admit analytical expansions which are omitted for brevity. In Eqs. (3)-(6) $\xi=\cosh u$ and $\mu=\cos\vartheta$ are prolate spheroidal coordinates in the notation of Nicholson [9] and I_ν denotes the modified Bessel function of the first kind with fractional order ν. Introducing Eqs. (3)-(6) allows expressing the diffraction component in prolate spheroidal coordinates. By analogy the incident wave potential is eventually written as

$$\phi_I^m = \frac{1}{K}\varepsilon_m i^m \sqrt{\frac{\pi}{2Kc}}\sum_{s=m}^{\infty}(-1)^{s-m}\,\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}\,\mathrm{I}_{s+1/2}(Kc)P_s^m(\mu)P_s^m(\xi) \tag{7}$$

The zero velocity condition on the wetted surface of the body results in a complex linear system of the form

$$F_m^m C_{ms}^m + \sum_{n=m+1}^{\infty} F_n^m C_{ns}^m = B_s^m \tag{8}$$

This system must be truncated to account for a finite number of modes M. The indices vary like m=0,1,…,M and n, s=m,m+1,…,M. Eq. (8) is solved M times for all orders m. For m=0, the elements C_{ms}^m and B_s^m compose a $N\times N$ complex matrix and a $N\times 1$ complex vector respectively, where N=M+1. For all subsequent orders the dimensions of the matrices are continuously reduced by one, whereas for the last order M+1 Eq. (8) becomes a simple linear algebraic equation. The coefficients C_{ms}^m and B_s^m are quite lengthy are their details are omitted. Finally the exciting forces are obtained by integrating the liner pressure $p = -\mathrm{i}\omega\rho(\phi_I + \phi_D)$ on the wetted surface of the spheroid.

3 Numerical results

Here the results of an indicative test case are presented. The geometry of the spheroid is defined by the axes ratio b/a=0.5. Also two immersion cases are considered, i.e. f/a=2.5 and 3.5. It should be mentioned herein that the analytical approach outlined above is only valid if the radius from the center of the spheroid is smaller than 2f (r<2f). Otherwise the method diverges, primarily due to Cauchy's principal value integral [2]. Nevertheless for a vertical prolate spheroid, even if it nearly touches the free surface, this restriction is fulfilled on the wetted surface where the linear pressure is required for calculating the exciting forces. The multipole expansion method is also able to provide the velocity potential at some distance from the body which unavoidably cannot be indefinitely large.

For validation purposes the present method calculations were compared against the respected macroelements approximation [10], according to which the velocity potential around the prolate spheroid is evaluated through matched eigenfunction expansions in properly defined ring-shaped fluid regions, which are obtained by approximating the body's meridian curve by a stepped curve. The comparisons are shown in Figs. 3-6 which depict the surge and heave exciting forces exerted on the body due to incoming regular waves. It is noted that the multipole expansion was truncated to account for M=17 modes. In fact that figure is considered conservative as the calculations converge satisfactorily quite faster. According to the depicted results the hydrodynamic loading is decreased for deeper placement of the spheroid. This is due to the fact that the velocity potential and accordingly the exciting forces are decreased exponentially with immersion. Also, for this particular case surge forces are larger than heave forces. In all cases examined herein the maximum of the transfer functions occurs at relatively small normalized wave frequency values Kb whereas the hydrodynamic loading tends asymptotically to zero for large Kb.

It is immediately evident that the convergence of the calculations obtained by the two methods is favorable. Especially for the surge forces the corresponding curves are nearly indistinguishable. Small differences are observed in the heave loading calculations where the macroelements approximation overestimates slightly the vertical forces. It is believed that these differences will vanish if the geometry of the prolate spheroid is going to be approximated using more ring-shaped cylindrical elements.

4 References

[1] Thorne RC. Multipole expansions in the theory of surface waves. Proc. Camb. Phil. Soc. 1953; 49:707-716.
[2] Wang S. Motions of a spherical submarine in waves. Ocean Engineering 1986; 13:249-271.
[3] Linton CM. Radiation and diffraction of water waves by a submerged sphere in finite depth. Ocean Engineering 1991; 18:61-74.
[4] Wu GX, Witz JA, Ma Q, Brown DT. Analysis of wave induced drift forces acting on a submerged sphere in finite water depth. Applied Ocean Research 1994; 16:353-361.

[5] Rahman M. Simulation of diffraction of ocean waves by a submerged sphere in finite depth. Applied Ocean Research 2001; 23:305-317.

[6] Havelock TH. Waves due to floating sphere making periodic heaving oscillations. Proc Royal Soc London 1955; A231: 1-7.

[7] Cooke JC. Some properties of Legendre functions. Proc Cambridge Philosophical Society 1953; 49:162-164

[8] Cooke JC. Some relations between Bessel and Legendre functions. Monatshefte für Mathematik 1956; 60:322-328

[9] Nicholson JW. Oblate Spheroidal Harmonics and their applications, Phil. Trans. R. Soc. London A 1924; 224, 49-93.

[10] Kokkinowrachos, K., Mavrakos, S.A., Azorakos, S. Behavior of vertical bodies of revolution in waves, Ocean Engineering, 13(6), 1986, 505-538.

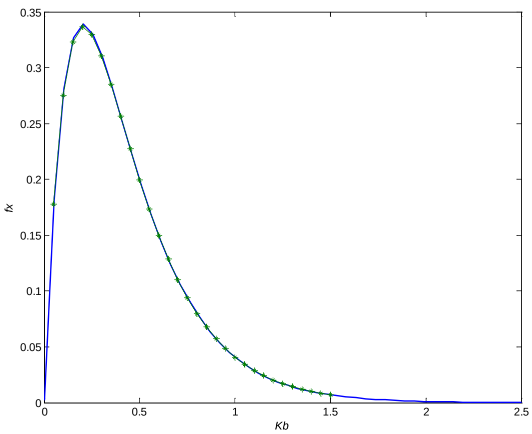

Fig. 3. Surge exciting forces on a prolate spheroid (b/a=0.5; f/a=2.5) normalized by $\pi\rho g A b^2$ (Lines: present method calculations; symbols: macroelements approximation).

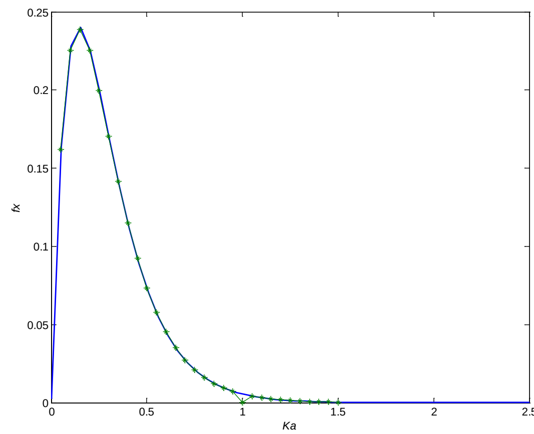

Fig. 5. Surge exciting forces on a prolate spheroid (b/a=0.5; f/a=3.5) normalized by $\pi\rho g A b^2$ (Lines: present method calculations; symbols: macroelements approximation).

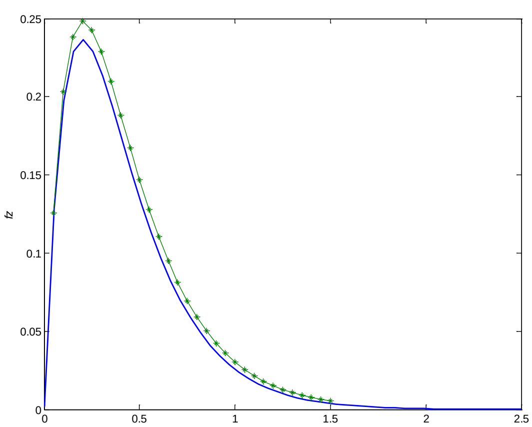

Fig. 4. Heave exciting forces on a prolate spheroid (b/a=0.5; f/a=2.5) normalized by $\pi\rho g A b^2$ (Lines: present method calculations; symbols: macroelements approximation).

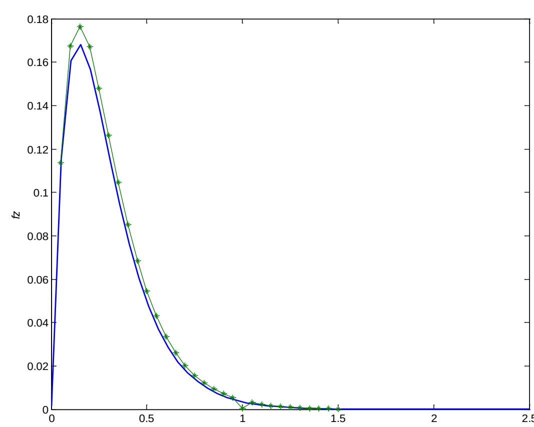

Fig. 6 Heave exciting forces on a prolate spheroid (b/a=0.5; f/a=3.5) normalized by $\pi\rho g A b^2$ (Lines: present method calculations; symbols: macroelements approximation).