Time-harmonic water waves trapped by surface-piercing motionless bodies floating freely

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We study the coupled time-harmonic motion of the mechanical system that consists of infinitely deep water, bounded above by a free surface, and a surface-piercing body floating freely. The surface tension is neglected and the water motion is irrotational, whereas the motion of the whole system is of small amplitude near equilibrium. The latter assumption allows us to apply a linear model; the coupling conditions are taken in the form proposed in [7] (see also [5], where the question of uniqueness is investigated when the depth of water is finite).

Our aim is to prove that for every value of frequency there exists a body (in fact, infinitely many bodies with axisymmetric submerged parts), which is motionless, but, nevertheless, traps waves. The corresponding trapped modes are similar to the *passive modes* found in [1] for a simplified model with a body constrained to the heave motion only. On the contrary, the motion of body is not restricted to any particular mode in the present work. Furthermore, the conditions guaranteeing the stability of the equilibrium position are considered in detail; they are subsidiary for the eigenvalue problem, but must hold for bodies floating freely. In the two-dimensional case, results similar to those presented here were obtained in [3].

1 Statements of the problem

Let the Cartesian coordinates $(\boldsymbol{x}, y)$, $\boldsymbol{x} = (x_1, x_2)$, be such that the y-axis is directed upwards, whereas the $\boldsymbol{x}$-plane coincides with the mean free surface. By $\widehat{B}$ we denote the domain occupied by the body in its equilibrium position (see fig. 1); $B = \widehat{B} \cap \mathbb{R}^3_-$ is the body's submerged part and $W = \mathbb{R}^3_- \setminus \overline{B}$ is the water domain, $\mathbb{R}^3_- = \{(\boldsymbol{x}, y) : \boldsymbol{x} \in \mathbb{R}^2, y < 0\}$. We suppose that W is simply connected, but B can consist of several connected components. Further notations are as follows: $\boldsymbol{n}$ is the normal on ∂W pointing to the exterior of W, $S = \partial \widehat{B} \cap \mathbb{R}^3_-$, $F = \partial W \setminus \overline{S}$, $D = \{\boldsymbol{x} \in \mathbb{R}^2,\ y = 0\} \setminus \overline{F}$ (see fig. 1).

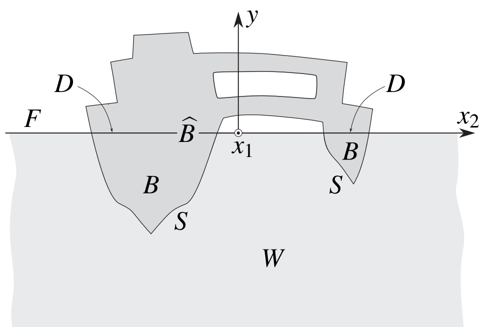

Figure 1: Definition sketch.

In the linearised setting, the motion of the whole system is described by the following first-order variables: the velocity potential $\Phi(\boldsymbol{x}, y; t)$ and the vector $\boldsymbol{q}(t) \in \mathbb{R}^6$, characterising the motion of the body's centre of mass about its rest position $\big(\boldsymbol{x}^{(0)}, y^{(0)}\big)$. The horizontal and vertical displacements are q_1, q_2 and q_4, respectively, whereas q_3 and q_5, q_6 are the angles of rotation about the axes that go through the centre of mass and are parallel to the y and x_1, x_2 axes, respectively. The time-dependent problem for Φ and $\boldsymbol{q}$ was obtained in [2] (see also [5, 7]), but we consider time-harmonic oscillations of the radian frequency $\omega > 0$, in which case $\big(\Phi(\boldsymbol{x}, y, t), \boldsymbol{q}(t)\big) = \mathrm{Re}\big\{\mathrm{e}^{-\mathrm{i}\omega t}\big(\varphi(\boldsymbol{x}, y), \mathrm{i}\boldsymbol{\chi}\big)\big\}$. Then the bounded complex-valued function φ and $\boldsymbol{\chi} \in \mathbb{C}^6$ must satisfy the problem:

$$\nabla^2 \varphi = 0 \ \text{ in } W; \quad \partial_y \varphi - \nu \varphi = 0 \ \text{ on } F, \ \text{ where } \nu = \omega^2 / g; \tag{1}$$

$$\partial_{\boldsymbol{n}} \varphi = \omega \boldsymbol{n}^{\mathsf{T}} \boldsymbol{D}_0 \boldsymbol{\chi} \ \text{ on } S; \quad \omega^2 \boldsymbol{E} \boldsymbol{\chi} = -\omega \int_S \varphi \boldsymbol{D}_0^{\mathsf{T}} \boldsymbol{n} \, \mathrm{d}s + g \boldsymbol{K} \boldsymbol{\chi}; \tag{2}$$

$$\int_{W \cap \{|\boldsymbol{x}| = a\}} \big|\partial_{|\boldsymbol{x}|} \varphi - \mathrm{i}\nu\varphi\big|^2 \mathrm{d}s = o(1) \ \text{ as } \ a \to \infty. \tag{3}$$

Here $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_y)$ is the spatial gradient and g is the acceleration due to gravity acting in the direction opposite to the y-axis; $\boldsymbol{D}_0 = \boldsymbol{D}\big(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\big)$, where $\boldsymbol{D}(\boldsymbol{x}, y) = \begin{bmatrix} 1 & 0 & x_2 & 0 & 0 & -y \\ 0 & 1 & -x_1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & -x_2 & x_1 \end{bmatrix}$; the matrix transposition is denoted by $^{\mathsf{T}}$. In the second condition (2) which is the equation of body's motion, we have two matrices defined as follows: $\boldsymbol{E} = \rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\, \boldsymbol{D}_0^{\mathsf{T}}(\boldsymbol{x}, y)\, \boldsymbol{D}_0(\boldsymbol{x}, y)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y$, where $\rho(\boldsymbol{x}, y) \geqslant 0$ is the distribution of density within the body and $\rho_0 > 0$ is the constant density of water;

$$\boldsymbol{K} = \begin{pmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{O}_3 & \boldsymbol{K}' \end{pmatrix}, \quad \text{where} \quad \boldsymbol{K}' = \begin{pmatrix} I^D & I_2^D & -I_1^D \\ I_2^D & I_{22}^D + I_y^B & -I_{12}^D \\ -I_1^D & -I_{12}^D & I_{11}^D + I_y^B \end{pmatrix}, \quad I^D = \int_D \mathrm{d}\boldsymbol{x}, \quad I_y^B = \int_B \big(y - y^{(0)}\big)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y,$$

$$I_i^D = \int_D \big(x_i - x_i^{(0)}\big)\, \mathrm{d}\boldsymbol{x}, \quad I_{ij}^D = \int_D \big(x_i - x_i^{(0)}\big)\big(x_j - x_j^{(0)}\big)\, \mathrm{d}\boldsymbol{x}, \quad i, j = 1, 2, \quad \text{and } \mathbb{O}_3 \text{ is the } 3\times 3 \text{ null matrix.}$$

The elements of $\boldsymbol{E}$ are various moments of the whole body $\widehat{B}$ (see [5]), and this 6×6 matrix is symmetric and positive definite. The matrix $\boldsymbol{K}$ related to buoyancy (see [2]) is symmetric.

Problem (1)–(3) must be augmented by conditions concerning the equilibrium position of the floating body: • $\rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y = \int_B \mathrm{d}\boldsymbol{x}\, \mathrm{d}y$ (Archimedes' law); • $\int_B \big(x_i - x_i^{(0)}\big)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y = 0$, $i = 1, 2$ (the center of buoyancy lies on the same vertical line as the centre of mass); • $\boldsymbol{K}'$ is a positive definite matrix (the body's equilibrium position is stable, [2, § 2.4]).

The boundedness of φ implies that $\nabla\varphi$ decays as $y \to -\infty$, whereas the radiation condition (3) guarantees that φ describes outgoing waves at infinity (see [4]). In the same way as in [5], one proves the following assertion about the energy of (φ, χ) satisfying (1)–(3):

The first component φ belongs to the space $H^1(W)$ and $\int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} < \infty$, that is, the kinetic and potential energy of the water motion is finite. The following equality expresses the equipartition of energy of the coupled motion

$$\int_W |\nabla\varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y + \omega^2 \overline{\chi}^{\mathsf{T}} \boldsymbol{E} \chi = \omega^2/g \int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} + g\, \overline{\chi}^{\mathsf{T}} \boldsymbol{K} \chi.$$

Thus, both the real and imaginary part of (φ, χ) are solutions of the problem, and so we formulate the following

Definition. Let the conditions concerning the equilibrium position hold, then a real, non-trivial pair (φ, χ), that belongs to $H^1(W) \times \mathbb{R}^6$, is called a *trapped mode* provided (1)–(2) are satisfied; the corresponding value of ω is a *trapping frequency*.

2 Modes trapped in axisymmetric water domains

In order to construct modes trapped by motionless surface-piercing bodies with axisymmetric immersed parts we apply the so-called semi-inverse procedure. Fixing $\omega > 0$, we seek an eigensolution in the form $(\varphi_*, \mathbf{0})$, where $\mathbf{0}$ is the null element of $\mathbb{R}^6$, that is, a corresponding trapping body is motionless. The subsidiary properties of the problem (Archimedes' law *etc.*) are satisfied through an appropriate choice of the density distribution $\rho(\boldsymbol{x}, y)$ for every surface-piercing trapping body $\widehat{B}$.

Velocity potential. According to the semi-inverse procedure, φ_* must be defined so that it has finite energy in any reasonable domain W, and relations (1) hold for φ_* in W. Let

$$\varphi_*(|\boldsymbol{x}|, y) = 2\int_0^\infty (k\cos ky + \nu\sin ky)\, I_\alpha(k|\boldsymbol{x}|)\, K_\beta(kr) \frac{k^2\, \mathrm{d}k}{k^2 + \nu^2} - \pi^2\nu^2 \mathrm{e}^{\nu y} J_\alpha(\nu|\boldsymbol{x}|)\, Y_\beta(\nu r), \; y \leqslant 0, \tag{4}$$

which is the real ring-dipole potential (cf. [1]), that generalises the two-dimensional potential proposed in [6]. Here $\alpha = 0$, $\beta = 1$ for $|\boldsymbol{x}| < r$ and $\alpha = 1$, $\beta = 0$ for $|\boldsymbol{x}| > r$, whereas $r > 0$ will be specified below; Y_0, Y_1 are the Neumann functions; J_0, J_1 and I_0, I_1, K_0, K_1 denote the standard and modified Bessel functions of the corresponding orders. It is easy to check that φ_* has a singularity when $|\boldsymbol{x}| = r$ and $y = 0$; moreover, φ_* is harmonic in $\mathbb{R}^3_-$, and the boundary condition $\partial_y\varphi_* - \nu\varphi_* = 0$ holds on $\partial\mathbb{R}^3_- \setminus \{|\boldsymbol{x}| = r, y = 0\}$.

We put $r = r_m = \nu^{-1} j_{1,m}$, where $j_{1,m}$ is one of the positive zeroes of J_1, and denote φ_* by φ_m when $r = r_m$. Our choice of r implies that the last term in (4) vanishes for $|\boldsymbol{x}| > r$ and $\varphi_m \in H^1(W)$ for any domain W obtained by removing some neighbourhood of the circle $\{|\boldsymbol{x}| = r_m, y = 0\}$ from $\mathbb{R}^3_-$. Therefore, any function φ_m can serve as the first component of the eigensolution provided the water domains W is chosen properly.

Stream functions. Since the second element of our eigensolution is $\mathbf{0}$, the second relation (2) means that the six integrals over S are equal to zero, which must be verified after constructing the surface S. Moreover, the right-hand side vanishes in the Neumann condition (the first relation (2)), where S must be chosen in accordance with the defined φ_*. Thus, it is convenient to use a Stokes stream function for finding admissible axisymmetric S. The stream function ψ is defined by φ through the following relations: $\partial_{|\boldsymbol{x}|}\varphi = -|\boldsymbol{x}|^{-1}\partial_y\psi$, $\partial_y\varphi = |\boldsymbol{x}|^{-1}\partial_{|\boldsymbol{x}|}\psi$. Using φ_m in these equations, we get that

$$\psi_m(|\boldsymbol{x}|,y) = -\pi^2\nu^2|\boldsymbol{x}|\mathrm{e}^{\nu y}J_1(\nu|\boldsymbol{x}|)Y_1(j_{1,m}) - 2|\boldsymbol{x}|\Psi(|\boldsymbol{x}|,r_m,y), \quad |\boldsymbol{x}| < r_m,\ y \leqslant 0,$$
$$\psi_m(|\boldsymbol{x}|,y) = -2|\boldsymbol{x}|\Psi(r_m,|\boldsymbol{x}|,y), \quad |\boldsymbol{x}| > r_m,\ y \leqslant 0,$$

where $\Psi(\sigma,\tau,y) = \displaystyle\int_0^\infty (k\sin ky - \nu\cos ky)\,I_1(k\sigma)\,K_1(k\tau)\frac{k^2\,\mathrm{d}k}{k^2+\nu^2}$ and the constant of integration is chosen so that $\psi_m(|\boldsymbol{x}|,y) \to 0$ as $|\boldsymbol{x}|^2 + y^2 \to \infty$.

According to equations for ϕ and ψ, we have that $\partial_{\boldsymbol{n}}\varphi = 0$ on every surface in $\mathbb{R}^3_-$, where $\psi = \mathrm{const}$, and so we list some properties of streamlines $\psi_m(|\boldsymbol{x}|,y) = \upsilon$ for various values of υ. Streamlines are smooth curves in $Q = \{|\boldsymbol{x}| > 0,\, y < 0\}$; their end-points belong to ∂Q for $\upsilon \neq 0$ and to $\partial Q \cup \{\infty\}$ for $\upsilon = 0$; a streamline emanates from every point on the half-axis $\{|\boldsymbol{x}| > 0,\, y = 0\}$ except those points, at which $\psi_m(|\boldsymbol{x}|,0)$ attains its local extrema, and the point $(r_m,0)$; notice that $\psi_m(|\boldsymbol{x}|,y) \to +\infty$ as $(|\boldsymbol{x}| - r_m)^2 + y^2 \to 0$ and $y \leqslant 0$.

3 Families of admissible water domains

Let $\upsilon \in \mathbb{R}$ be such that $S_{\upsilon,m} = \{(\boldsymbol{x},y) : \psi_m(|\boldsymbol{x}|,y) = \upsilon,\, y < 0\}$ is a bounded surface which divides $\mathbb{R}^3_-$ into two domains so that one of them, say, $B_{\upsilon,m}$ is bounded and has the circumference $\{|\boldsymbol{x}| = r_m,\, y = 0\}$ inside the upper flat part of its boundary. Then $W_{\upsilon,m} = \mathbb{R}^3_- \setminus \overline{B_{\upsilon,m}}$ can be taken as an axisymmetric water domain. A family of so defined domains exists for every $m \geqslant 1$ and corresponds to positive levels of $\psi_m(|\boldsymbol{x}|,y)$. These level lines emanate from the half-axis $|\boldsymbol{x}| > r_m$ and have their second end-points on the interval $(0,r_m)$ of the $|\boldsymbol{x}|$-axis (see figure 2 (b)).

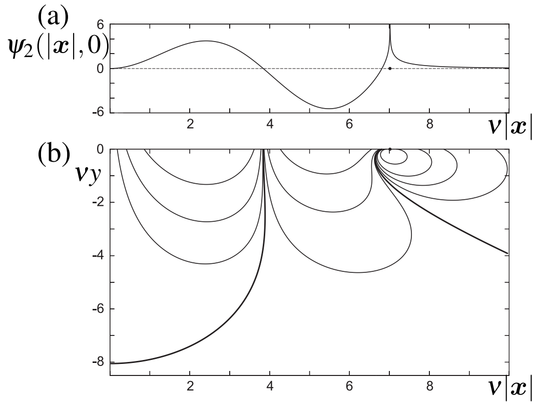

Figure 2: (a) The trace $\psi_2(|\boldsymbol{x}|,0)$. (b) Streamlines $\psi_2(|\boldsymbol{x}|,y) = \upsilon$ for various values of υ; nodal lines ($\upsilon = 0$) are plotted with bold lines.

It is straightforward to check that $(\varphi_m, \mathbf{0})$ is an eigensolution in $W_{\upsilon,m}$. This family of domains is the simplest one; other families are defined by two or more parameters. In particular, a family of admissible water domains is defined by ψ_1 and is parameterised by pairs (υ_+,υ_-), where $\upsilon_+ > 0$ and $\upsilon_- \in (\check{M},0)$. (The negative minimum $\check{M}$ of $\psi_1(|\boldsymbol{x}|,0)$ is attained for some $|\check{\boldsymbol{x}}| < r_1$.) The corresponding water domain $W_{\upsilon_+,\upsilon_-}$ is the complement in $\mathbb{R}^3_-$ of two disconnected toroidal bodies $\overline{B_{\upsilon_\pm}} = \{(\boldsymbol{x},y) : \pm\psi_1(|\boldsymbol{x}|,y) \geqslant \pm\upsilon_\pm,\, y \leqslant 0\}$.

4 Examples of trapping bodies

We describe three families of trapping bodies. The simplest one consists of bodies comprising of a single toroid; double-toroid bodies form the second family; the last family consists of bodies whose shape suggests to call them 'dummy-like'.

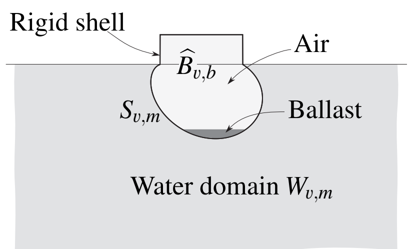

Figure 3: The cross-section of a single-toroid trapping body by a half-plane adjacent to the y-axis.

In order to obtain a trapping body belonging to the first family, one takes a domain $B_{\upsilon,m}$ described in § 3 and complements it by the above-water part that has a rectangular vertical cross-section of the height b (see fig. 3). (Of course, other above-water parts are admissible; they can have arbitrary shape symmetric about the x_1 and x_2 axes.) The constructed body is formed by the rigid shell $\partial\widehat{B}_{\upsilon,b}$, that encloses air and a ballast layer of constant density at the bottom. Taking the density of ballast to be sufficiently large and determining the thickness of the ballast layer from Archimedes' law, one gets that the position of the centre of mass of $\widehat{B}_{\upsilon,b}$ (by symmetry it is on the y-axis) is as close to the lowest level of $B_{\upsilon,m}$ as one pleases. Therefore, the last two subsidiary conditions hold as well (in fact, $\boldsymbol{K}'$ is a diagonal matrix with positive elements). Thus the following assertion holds.

Let ω be an arbitrary positive number. Then $\widehat{B}_{\upsilon,b}$ with $\upsilon, b > 0$ and masses distributed inside it as described above is a trapping body floating freely and the corresponding eigensolution of the coupled problem is $(\varphi_m,\mathbf{0})$, where $\varphi_m \in H^1(W_{\upsilon,m})$ is defined by formulae (4) *with $r = r_m$.*

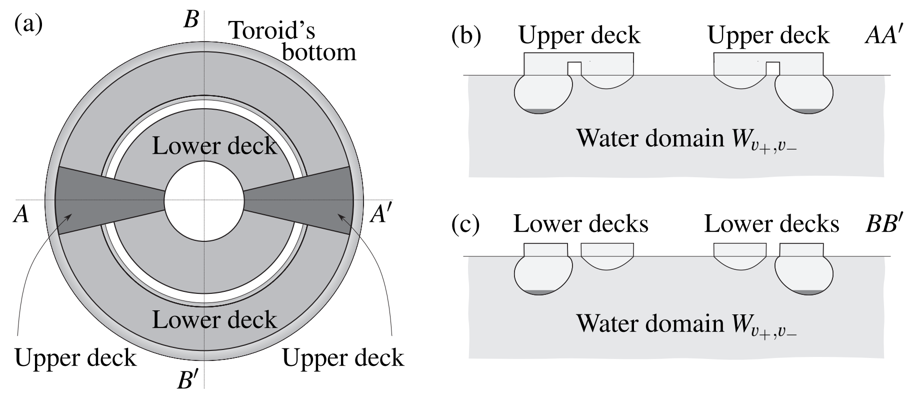

Figure 4: A trapping body that consists of two clamped toroids. Top view (a); vertical cross-sections along AA' (b) and along BB' (c). At the bottom of the exterior toroid, the ballast layer is shown in (b) and (c).

In the same way one obtains a trapping body that consists of two clamped toroids. Let $m = 1$ and let B_{υ_+} and B_{υ_-} correspond to $\upsilon_+ > 0$ and $\upsilon_- \in (\breve{M}, 0)$, respectively. Finally, let (see fig. 4)

$$\widehat{B}_{\circledcirc} = \left\{(\boldsymbol{x}, y) : |\boldsymbol{x}| \in \left(r_{\text{int}}^{(-)}, r_{\text{ext}}^{(-)}\right) \cup \left(r_{\text{int}}^{(+)}, r_{\text{ext}}^{(+)}\right), 0 \leqslant y < b_1\right\} \cup B_{\upsilon_-} \cup B_{\upsilon_+} \cup \Sigma_{b_2},$$

where $r_{\text{int}}^{(\pm)}$ and $r_{\text{ext}}^{(\pm)}$ are defined by $\psi_1\left(r_{\text{int}}^{(\pm)}, 0\right) = \psi_1\left(r_{\text{ext}}^{(\pm)}, 0\right) = \upsilon_\pm$, and Σ_{b_2} describes the two clamps of height b_2, connecting toroids. Then the body's constituents are the rigid shell $\partial\widehat{B}_{\circledcirc}$, which encloses air and a ballast layer of constant density at the bottom. The same considerations as above lead to the analogous assertion, guaranteeing that $\widehat{B}_{\circledcirc}$ is a trapping structure.

In order to get the 'dummy-like' trapping body one has to apply the following choice of two level surfaces of ψ_2. One of them is the nodal surface that crosses the y-axis (the corresponding nodal line is plotted bold in figure 2); the second surface is one of those that generate the family $\{S_{\upsilon,2}\}$. Clamping the nodal surface and that belonging to $\{S_{\upsilon,2}\}$, one obtains a dummy-like trapping body (an example of such a body is shown in fig. 5).

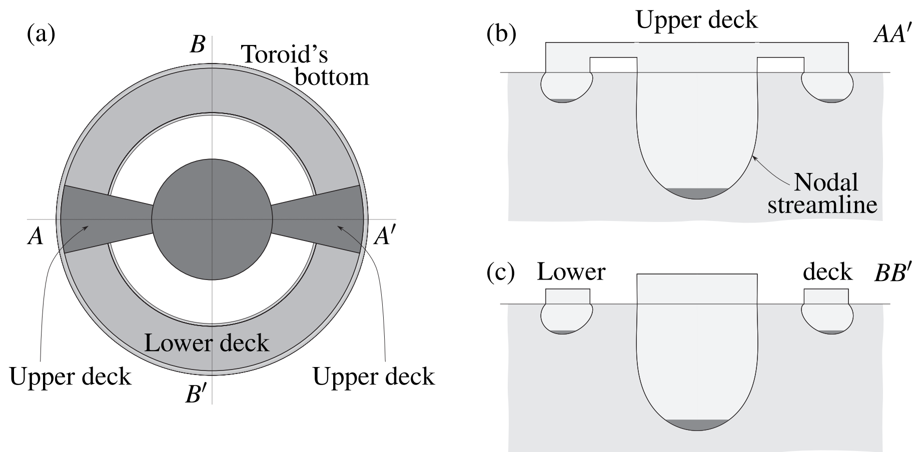

Figure 5: A dummy-like trapping body that consists of a toroid clamped to the nodal body. Top view (a); vertical cross-sections along AA' (b) and along BB' (c). The ballast layers are shown at the bottom in (b) and (c).

5 Conclusion and discussion

Examples of freely floating bodies that are motionless, have axisymmetric immersed parts and trap time-harmonic water waves are constructed by means of the semi-inverse procedure. As in the two-dimensional case [3], the existence of trapped modes is proved without any restriction on the mode of body's motion. It is still an open question whether the existence of a non-motionless body, that floats freely and traps time-harmonic waves, can be rigorously proved. It is worth mentioning that the motionless trapped modes also serve as eigensolutions for the water wave problem in fixed domains.

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