# Unsteady motion of elliptic cylinder under ice cover

**Vasily K. Kostikov[1], Nikolay I. Makarenko[1], Alexander A. Korobkin[1,2]**
**[1]Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia**
**[2]School of Mathematics, University of East Anglia, Norwich, UK**
**E-mail: vasilii_kostikov@mail.ru**

Nonlinear problem of unsteady hydroelastic waves generated by a submerged elliptic cylinder moving under ice sheet is studied by theoretical means. Original equations of hydrodynamics and ice response are reduced to a system of integro-differential equations. Small-time asymptotic solution of the system is obtained for a cylinder moving from rest with a constant acceleration. Initial stage of impulsive free-surface flow forced by a moving circular cylinder was studied by Tyvand & Miloh (1995a,b). In the present analysis we use the boundary equation technique developed by Makarenko (2003, 2004) for unsteady problems of wave-body interaction.

## 1 Formulation of the problem

Two-dimensional potential flow of infinitely deep water is considered in the Cartesian coordinate system $Oxy$ (see Figure 1). The fluid is covered with a flexible ice sheet of thickness $h$. Initially the fluid is at rest. The line $y = 0$ corresponds to the undisturbed interface between the ice and fluid. Non-dimensional variables are used below. Initial distance $h_0$ of the cylinder center from the ice sheet is taken as the length scale of the problem and a characteristic speed of the body $u_0$ as velocity scale of the flow. Elastic deflection of the ice sheet $\Gamma(t)$ is described by the equation $y = \eta(x,t)$, where $\eta(x,0) = 0$. The ice sheet is modeled as an infinite Euler beam with the deflection $\eta(x,t)$ decaying at the infinity. The elliptic cylinder moves without rotation. The non-dimensional coordinates of the center of the cylinder are given as $x = x_{cyl}(t)$ and $y = y_{cyl}(t)$, where $y_{cyl}(0) = -1$. In Figure 1, $a$ and $b$ are the non-dimensional semi-axes of the cylinder, $a \geq b$.

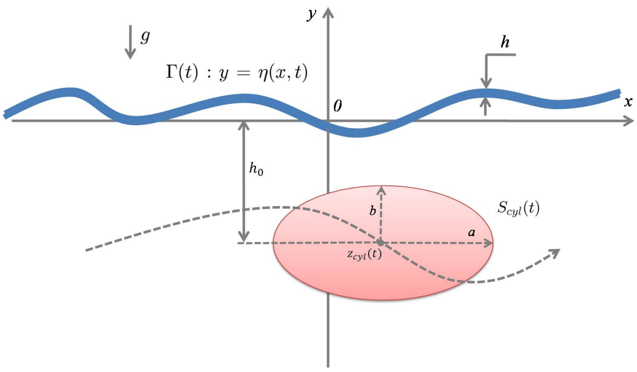


Fig 1: Scheme of motion

Unsteady flow of incompressible fluid is described by the Euler equations for the velocity of the flow $\mathbf{u} = (U, V)$ and hydrodynamic pressure $p$

$$\begin{cases} U_t + UU_x + VU_y + p_x = 0, \\ V_t + UV_x + VV_y + p_y = -\lambda, \\ U_x + V_y = 0, \quad U_y - V_x = 0, \end{cases} \tag{1}$$

where $\lambda = gh_0/u_0^2$ is the square of the inverse Froude number and $g$ is the gravity acceleration.

Boundary conditions on the ice-fluid interface $\Gamma(t)$ are

$$\eta_t + U\eta_x = V, \quad p = \alpha\eta_{tt} + \beta\eta_{xxxx} \qquad (y = \eta(x,t)). \tag{2}$$

The parameter $\alpha = \rho_{ice}h/(\rho h_0)$ characterizes the inertia of the ice sheet, $\rho$ is the fluid density, and the parameter $\beta = Eh^3/(12\rho h_0^3 u_0^2)$ is the flexural rigidity of the ice sheet. In the limit $\beta = 0$, we obtain the model of broken ice. If both $\alpha = 0$ and $\beta = 0$ in the dynamic condition (2), we arrive at the problem of a body motion beneath the liquid free surface without ice.

The boundary condition on the cylinder surface $S_{cyl}(t)$ has the form

$$(\mathbf{u} - \mathbf{u_{cyl}}) \cdot \mathbf{n} = 0, \quad (x,y) \in S_{cyl}(t), \tag{3}$$

where $\mathbf{n}$ is the unit normal. We assume that the fluid is at rest at infinity, so we have $U, V, \eta \to 0$ as $|\mathbf{x}| \to \infty$. We assume also that the initial velocity field $\mathbf{u}_{|t=0} = \mathbf{u}_0$ satisfies the compatibility conditions

$$U_{0x} + V_{0y} = 0, \quad U_{0y} - V_{0x} = 0,$$
$$(\mathbf{u}_0 - \mathbf{u_{cyl}}(0)) \cdot \mathbf{n}_0 = 0, \quad (x,y) \in S_{cyl}(0).$$

These conditions are satisfied, in particular, when the cylinder starts moving smoothly from rest.

## 2 Iintegro-differential equations at the interface

Let us introduce the tangential velocity $u(x,t) = (U + \eta_x V)_{|y=\eta}$ and normal velocity $v(x,t) = (V - \eta_x U)_{|y=\eta}$ of the fluid particles which are in contact with the interface $\Gamma(t)$. The equations (1)-(2) can be reduced to an equivalent system of one-dimensional equations for the unknown functions $u, v, \eta$:

$$\eta_t = v, \qquad u_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{u^2 - 2\eta_x uv - v^2}{1+\eta_x^2}\right) + \lambda\eta_x + \alpha\eta_{ttx} + \beta\eta_{xxxxx} = 0. \tag{4}$$

The differential equations (4) should be considered together with the boundary integral equation

$$\pi v(x) + v.p.\int_{-\infty}^{\infty} A(x,s)\, v(s)\, ds = v.p.\int_{-\infty}^{+\infty} B(x,s)\, u(s)\, ds + v_{dip}(x), \tag{5}$$

which is the boundary equation for the analytic function $F(z,t) = U - iV$, where $z = x + iy$. The time $t$ is not shown in (5) because it plays the role of a parameter in this equation. The kernels $A$ and $B$ can be split as

$$A = A_f + r^2 A_r + c^2 A_c, \qquad B = B_f + r^2 B_r + c^2 B_c$$

where $r = (a+b)/2$ and $c = \sqrt{a^2 - b^2}$ is proportional to the eccentricity of the body contour $S_{cyl}$. The kernels $A_f$ and $B_f$ depend on the deflection $\eta$ as

$$A_f(x,s) + iB_f(x,s) = \frac{i[1 + i\eta_x(x)]}{x - s + i[\eta(x) - \eta(s)]}.$$

Note that the simplified version of the the system (4)–(5) with truncated kernels $A = A_f$ and $B = B_f$ in (5) and $v_{dip} = 0$ corresponds to the model of hydroelastic waves (1)—(2) without any submerged body. Correspondingly, the kernels $A_r, A_c, B_r$ and $B_c$ describe the ice-body interaction. To define these kernels, we introduce an auxiliary complex-valued function

$$\tau(z) = \begin{cases} \dfrac{1}{2}\left(z - z_{cyl} + \sqrt{(z - z_{cyl})^2 - c^2}\right), & -\dfrac{\pi}{2} < Arg(z) < \dfrac{\pi}{2}, \\[2ex] \dfrac{1}{2}\left(z - z_{cyl} - \sqrt{(z - z_{cyl})^2 - c^2}\right), & \dfrac{\pi}{2} < Arg(z) < \dfrac{3\pi}{2}, \end{cases}$$

where $z_{cyl}(t) = x_{cyl}(t) + iy_{cyl}(t)$. The function $\tau(z)$ presents analytic branches of inverse mapping by the Joukowski function $z(\tau) = z_{cyl} + \tau + c^2/(4\tau)$ which maps the exterior of the ellipse with semi-axes $a$ and $b$ onto the exterior $|\tau| \geqslant r$ of the circle of radius $r = (a+b)/2$. By using this mapping we can avoid the integrals along the body contour in (5) by transforming them to the integrals along the interface $\Gamma$. The kernels $A_r$ and $B_r$ are given as

$$A_r(x,s) + iB_r(x,s) = \frac{1}{\tau(s+i\eta(s)) - \overline{\tau_*(x+i\eta(x))}} \left[\frac{i}{\tau(x+i\eta(x))}\right]'_x,$$

and the kernels $A_c$ and $B_c$ are given by the formula

$$A_c(x,s) + iB_c(x,s) = \frac{1}{4r^2\tau(s+i\eta(s)) - c^2\overline{\tau_*(x+i\eta(x))}} \left[\frac{ir^2}{\tau(x+i\eta(x))}\right]'_x$$

where bar denotes complex conjugate, and $\tau_* = r^2/\overline{\tau}$. The function $v_{dip}$ in the right-hand side of (5) is given by

$$v_{dip}(x) = Re\left[\frac{\pi i\left\{c^2\overline{z'_{cyl}(t)} - 4r^2 z'_{cyl}(t)\right\}}{2\tau(x+i\eta(x))}\right]'_x.$$

The velocity $v_{dip}$ is a normal fluid velocity induced at $\Gamma$ by conformal image of dipole concentrated at the origin $\tau = 0$ of auxiliary $\tau$-plane. The integro-differential system (4), (5) is equivalent to the original equations (1)–(3).

# 3 Small-time asymptotic solution

We consider the unsteady flow which starts from rest, $\eta(x,0) = u(x,0) = v(x,0) = 0$, and is caused by the motion of the cylinder with a constant acceleration. The solution of equations (4)-(5) is derived by the small-time expansion method in the form

$$\begin{aligned}
\eta(x,t) &= t^2\eta_2(x) + t^3\eta_3(x) + t^4\eta_4(x) + \dots, \\
u(x,t) &= tu_1(x) + t^2u_2(x) + t^3u_3(x)\dots, \\
v(x,t) &= tv_1(x) + t^2v_2(x) + t^3v_3(x) + \dots.
\end{aligned}$$

The coefficients $\eta_n$ and $u_n$ are evaluated explicitly by recursive formulae which follow from the differential equations (4) once the coefficients $v_n$ are known. The integral equation (5) provides recursive equations for the coefficients $v_n(x)$ of the form

$$\pi v_n(x) + \int_{-\infty}^{+\infty} A_0(x,s)\, v_n(s)ds = \varphi_n(x), \quad (n \geqslant 1)$$

where the kernel $A_0$ consists of the lower-order terms of the kernels $A$ and $B$ with respect to the deflection $\eta$. Further simplifications are available by using a perturbation procedure with some appropriate small parameters. If the nondimensional semi-axes $a$ and $b$ are small, we obtain the non-linear version of dipole approximation for submerged elliptic cylinder. Combining this approach with the method of successive approximations and assuming that the parameters $\alpha$ and $\beta$ are small, one can obtain analytically explicit formulae for the leading-order solution. For instance, we consider here symmetric wave patterns forming by vertical motion of the cylinder. In this case, the trajectory of the center can be taken in dimensionless form as $x_{cyl}(t) = 0$, $y_{cyl}(t) = -1 - t^2$. The leading-order solution has the following coefficients for the ice deflection $\eta$:

$$\eta_2(x) = -a(a+b)Q'(x;c), \quad \eta_3(x) = 0,$$
$$\eta_4(x) = -\frac{a(a+b)}{2}\left(1+\frac{\lambda}{6}\right)\left(P''(x;c) - \alpha Q'''(x;c)\right) - \frac{\beta a(a+b)}{12}P^{VI}(x;c). \tag{6}$$

Here $P(x;c)$ is even and $Q(x;c)$ is odd functions of $x$. These functions are defined for $x \geqslant 0$ by the formulae

$$P(x;c) = -\frac{2}{c^2} + \frac{\sqrt{2}}{c^2}\sqrt{\sqrt{(x^2-c^2-1)^2+4x^2} - x^2 + c^2 + 1},$$
$$Q(x;c) = \frac{2x}{c^2} - \frac{\sqrt{2}}{c^2}\sqrt{x^2-c^2-1+\sqrt{(x^2-c^2-1)^2+4x^2}}.$$

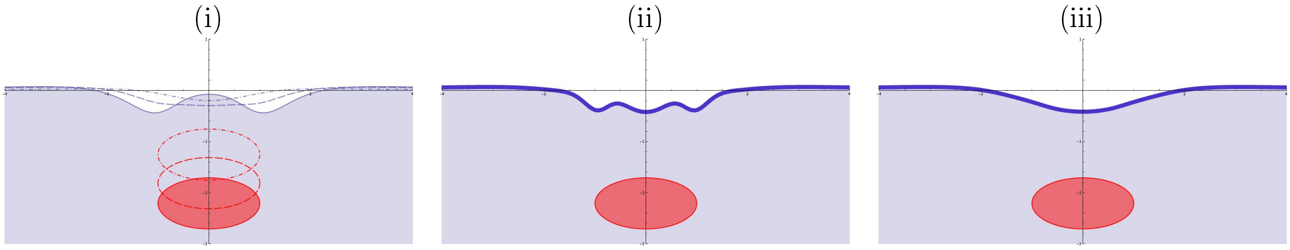


Fig 2: Vertical motion of the elliptic cylinder with semi-axes $a = 1$, $b = 0.5$. (i) Free surface flow ($\alpha = 0$, $\beta = 0$). Three successive time instants $t_1 = 0.5$, $t_2 = 0.9$, $t_3 = 1.1$. (ii) Elastic deflection of the ice sheet ($\alpha = 0.15$, $\beta = 0.05$) at $t = 1.1$. (iii) Elevation of the broken ice surface ($\alpha = 0.15$ , $\beta = 0$) at $t = 1.1$.

The flow regimes related to the solution (6) are shown in Figure 2 which demonstrates significant difference between the interface shapes (i)-(iii) depending on the parameters of ice cover. It is easy to see that the splash jet formed in the case (i) on the free surface during the early stage is strongly suppressed by floating ice. At the same time, in the case (ii), the ice sheet conducts also short-crested bending waves generated due to local disturbance of the flow caused by the cylinder, and these elastic waves do not appear in the case (iii) of broken ice.

## 4 Acknowledgment


This work was supported by the Federal Program RTPIR "Mechanics and thermodynamics of ice cover" (grant No 14.740.11.0878).


## 5 Refereences


Makarenko N.I., 2003. Nonlinear interaction of submerged cylinder with free surface. JOMAE Transactions of the ASME. Vol.125. No 1, pp. 72-75.

Makarenko N.I., 2004. Nonlinear water waves in the presence of submerged elliptic cylinder. Proc. 23rd Int. Conf. on Offshore Mech. and Arctic Engineering OMAE'04. June 20-25, 2004, Vancouver, Canada. Paper OMAE-51413.

Tyvand P.A., Miloh T., 1995a. Free-Surface Flow due to Impulsive Motion of a Small Submerged Circular Cylinder. J. Fluid Mech. Vol. 286, pp. 67-101.

Tyvand P.A., Miloh T., 1995b. Free-Surface Flow Generated by a Small Submerged Circular Cylinder Starting from the Rest. J. Fluid Mech. Vol. 286, pp. 103-116.