# Nonlinear Forces on a Submerged, Horizontal Plate: The G-N Theory *

M. Hayatdavoodi and R.C. Ertekin

Department of Ocean and Resources Engineering,
University of Hawaii, Honolulu, Hawaii 96822, USA
Eml: masoud@hawaii.edu ertekin@hawaii.edu

**Abstract**

Propagation of nonlinear waves of solitary and cnoidal type over a submerged horizontal, fixed flat plate is studied. The nonlinear and unsteady Green-Naghdi equations (Level I) are solved to estimate the pressure distribution on top and bottom of the plate and the wave-induced vertical and horizontal forces. Results are compared with the available laboratory experiments. A parametric study is conducted to find an expression for the shallow water wave-induced loads on the structure based on the structure geometry and wave characteristics.

## Introduction

Coastal bridges, piers, jetties and decks are all among costal structures which are in danger of becoming fully submerged due to the storm surge and wave set up during a storm or hurricane event. Submerged plates and cylinders are also used as breakwaters or wave energy converters. These structures are mostly located in shallow water and coastal areas, where the size of the structure is not necessarily small compared to the wavelength. Approaches similar to that of Morison et al. (1950) or Kaplan (1992) and other empirical relations may be suitable for deep waters, where the size of the structure compared to wavelength can be considered small.

Wave-induced loads and moments on a submerged plate or a rectangular box in shallow water is studied by a direct approach, by use of the Green-Naghdi (G-N hereafter) equations. First, the equations are introduced briefly followed by the application of the theory to the problem of wave-induced loads on the submerged plate. This is following by the presentation of the results.

## Green-Naghdi Equations

An alternative to the perturbation methods commonly used in shallow-water wave hydrodynamics is the direct approach first established by A.E. Green and P.M. Naghdi in the early 1970s. In this method, which is originally based on the theory of shells and plates in structural mechanics, the continuum is modeled by a directed or Cosserat surface. The theory of directed fluid sheet, associates inertia, linear momentum and angular momentum to the Cosserat surface and obtains a general set of dynamical equations to explain the deformations of a sheet-like body. A derivation of the equations can be found in Green & Naghdi (1976). The governing equations for wave propagation in water of variable depth are coined the Green-Naghdi equations by Ertekin (1984) who has studied a number of constrained domain problems in shallow water involving solitons. The G-N equations satisfy the constitutive laws (conservation of mass, linear momentum, angular momentum and director momentum) and the free-surface and sea-floor boundary conditions exactly. They are categorized according to their Levels, each increasing Level corresponding to an increasing number of governing equations, and in principle to increasing level of accuracy. The G-N equations were derived originally without making the assumption of irrotational flow, however, such an assumption can be made in deriving the irrotational version of the G-N equations for any Level, see Kim et al. (2001).

The G-N equations are used here in their two-dimensional form. A right-handed coordinate system with $x$ pointing to the right and $z$ directed against gravity is considered. The free surface, $\eta(x,t)$, is measured from the still-water level, and $h$ is the constant water depth. The free surface pressure is denoted by $\hat{p}(x,t)$. In the Level I G-N equations, it is assumed that the vertical velocity, $w(x,t)$, varies linearly and the horizontal

*This work is supported by the Link Foundation Ocean Engineering and Instrumentation Fellowship to M.H., and by the State of Hawaii Department Of Transportation, Award No. DOT-08-004, TA 2009-1R.

velocity, $u(x,t)$, is constant along the water column. The Level I equations are given by the mass and combined momentum equations:

$$\eta_{,t} + (h + \eta - \alpha)u_x = \alpha_t \, , \tag{1}$$

$$\dot{u} + g\eta_x + \frac{\hat{p}_x}{\rho} = -\frac{1}{6}[2\eta + \alpha]_x\ddot{\alpha} + [4\eta - \alpha]_x\ddot{\eta} + (h + \eta - \alpha)[\ddot{\alpha} + 2\ddot{\eta}]_x \, , \tag{2}$$

where $\alpha(x,t)$ is the vertical location of the bottom of the fluid sheet, $\rho$ is the mass density of the fluid, $g$ is the gravitational acceleration, and superposed dot is the two-dimensional material time derivative. The three dimensional version of the equations can be found in Ertekin et al. (1986). Solitary and periodic solutions of Eqs. (1) and (2) can be found in Ertekin (1984) and Ertekin & Becker (1998), for example.

The spatial derivatives of the bottom profile, $\alpha$, in the above equations do not allow for the equations to be applied when there is a discontinuity on the seafloor. Instead, one can divide the domain into different regions and solve the equations separately in each region. This domain separation, however, causes a sudden change in mass, momentum and energy through the discontinuity line. As discussed by Green & Naghdi (1976), to obtain a valid solution for the entire domain, appropriate jump conditions must be used as we will do here.

## Application

The problem of nonlinear waves propagating over the submerged plate is studied by dividing the domain into four different regions, shown in Fig. 1. The solution for the entire domain, $-\infty < x < +\infty$, is obtained by finding the solutions for each region and matching them at the leading ($x = x_L$) and trailing ($x = x_T$) edges of the plate, where the regions meet, using the jump and matching conditions demanded by the theory and the physics of the problem.

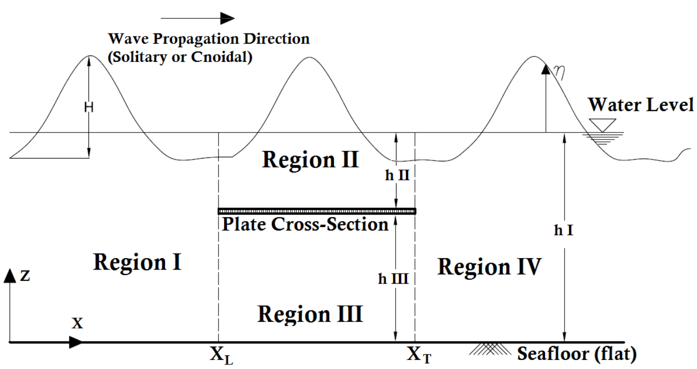


Figure 1: A sketch of nonlinear waves propagate over a submerged plate, showing the four regions referred to in the text.

In Regions I, II and IV, the top surface is free and the top pressure, $\hat{p}$, is equal to the constant atmospheric pressure, $p_0$. The depth of the fluid sheet is constant in these regions and is equal to $h_I$ in Regions I and IV and $h_{II}$ in Region II. In Region III, water depth is constant $h_{III}$, the top surface is specified and the top pressure, $\hat{p}_{III}$, is unknown. The G-N equations predict the horizontal velocity in Region III to be only a function of time and constant in space, and the vertical component of velocity is always zero. The equations of motion in all regions are solved simultaneously.

At the leading and trailing edges of the plate, the theory demands for conservation of mass and linear momentum in horizontal direction through the discontinuity line. These two conditions lead to two jump conditions namely conservation of mass and linear momentum in the horizontal direction. The final form of the jump conditions are

$$[[\rho\phi u]] = 0 \quad \text{and} \quad \left[\left[\rho\phi u^2 + p\right]\right] = \lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}(\hat{p}\phi_x)dx, \tag{3}$$

where $\phi$ is the thickness of the fluid sheet ($\phi = \beta - \alpha$), $\beta$ is the vertical location of the top fluid surface and $p$ is the integrated pressure along the fluid column. The notation $[[\quad]]$ used in the above equations is defined as

$$[[f]] = f^+ - f^- \quad \text{where} \quad f^+ = \lim_{x\to x_0+\delta} f \qquad \text{and} \quad f^- = \lim_{x\to x_0-\delta} f. \tag{4}$$

The jump condition associated with the conservation of energy is not applicable to this problem as the fluid is inviscid and every other dissipation mechanisms are not present. This will be discussed elsewhere.

In addition to the jump conditions demanded by the theory, the following physical conditions are required as the matching conditions at the leading and trailing edges

$$[[\eta]] = 0 \quad \text{at} \quad z = h + \eta \quad \text{and} \quad x = x_L \quad \& \quad x = x_T, \tag{5}$$

$$[[\bar{p}]] = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad x = x_L \quad \& \quad x = x_T, \tag{6}$$

$$\hat{p}_{III} = \bar{p}_{II} \quad \text{at} \quad z = h_{III} \quad \text{and} \quad x = x_L \quad \& \quad x = x_T, \tag{7}$$

$$[[p]] = 0 \quad \text{at} \quad x = x_L \quad \& \quad x = x_T, \tag{8}$$

where, $\bar{p}$ is the fluid pressure at the bottom surface in each region. Other variables are defined in the previous equations. Physically, Eqs. (5)-(7) express continuous surface elevation, continuous bottom pressure and continuous local pressure along the discontinuity line. Eq. (8) enforces the same integrated pressure before and after the discontinuity lines at the leading and trailing edges of the plate. To be specific, Eq. (8) can be written as

$$p_I = p_{II} + p_{III} \quad ,\text{at } x = x_L \quad \text{and} \quad p_{IV} = p_{II} + p_{III} \quad ,\text{at } x = x_T, \tag{9}$$

where $p_i$ is the integrated pressure in the region specified by index i.

## Results and Discussion

The G-N equations are solved numerically by the finite difference method, second-order accurate in space, and with the modified Euler method for time integration. Fig. 2 shows the diffraction of a solitary wave due to the presence of the submerged plate. The reflected wave is a negative solitary wave with the amplitude of about 0.12 times of the incident wave. This is observed in wave gauge I. The solitary wave is about to split into two waves when on top of the plate (wave gauge II), and the fully developed soliton fission is observed at wave gauge III. Pressure on the top and bottom of the plate can be calculated solving the G-N equations and is shown in Fig. 3, at the middle of the plate. The top pressure increases to a maximum value and then decreases to the initial value (hydrostatic pressure) as the wave passes. The bottom pressure, however, has an initial increase right before the wave arrives to the point. The magnitude of the initial hump is comparable to the magnitude of the top pressure. Then, the bottom pressure decreases to a minimum value just before the crest of the wave arrives. As the wave crest reaches the point, the bottom pressure increases to its maximum value within a very short duration. With some oscillations, the bottom pressure reduces to the initial value as the wave back passes. The vertical wave induced force is computed by integrating the bottom-top pressure differential along the plate width at each time step. The horizontal force per unit thickness of the plate is computed by knowing the pressure differential at the leading and trailing edges of the plate at each time step.

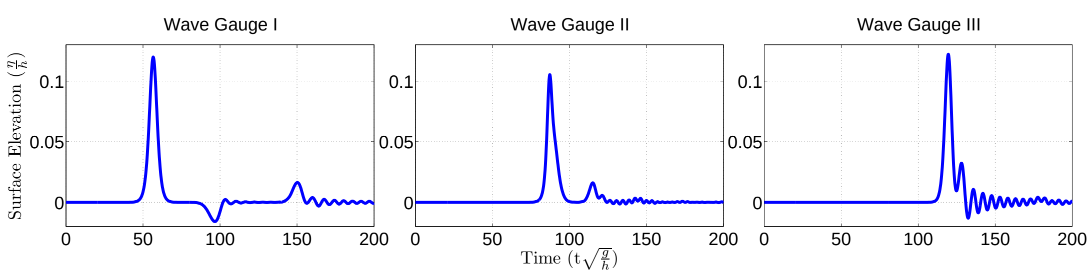


Figure 2: Diffraction of a solitary wave passing over a submerged plate. Wave amplitude= 0.12h; Plate width= 20h; Plate submergence depth= 0.5h. Wave gauge I is located at one plate width upstream. Wave gauge II is at the middle of the plate and wave gauge III is located a plate width downstream.

Results are compared with the available laboratory experiments. Fig.4 shows the comparison of the vertical and horizontal forces due to the propagation of periodic waves over a submerged cylinder conducted by Brater et al. (1958). Cnoidal waves are modeled by the G-N equations. A good agreement is observed with the experimental data.

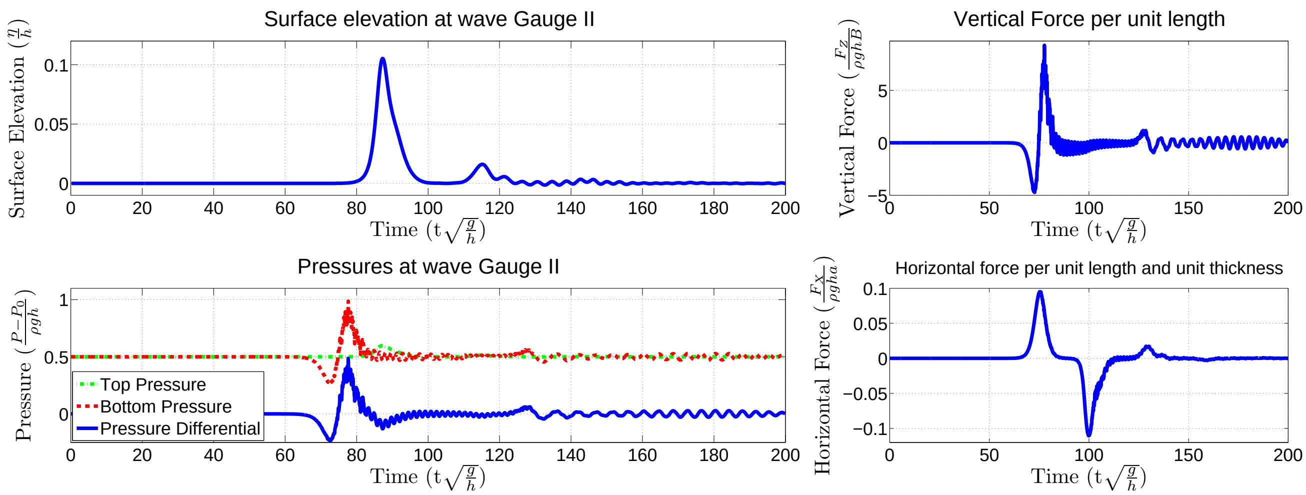


Figure 3: Computed time history of the top, bottom and the differential pressures at wave gauge II, and the total dimensionless vertical and horizontal forces. Pressures are gauge pressure (compared with the constant atmospheric pressure). Wave and plate characteristics are the same as in Fig 2.

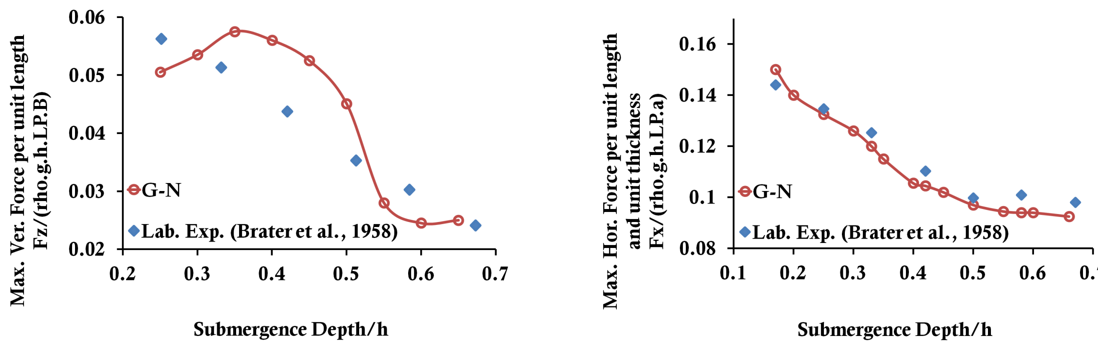


Figure 4: Vertical and horizontal forces on a submerged cylinder. Wave characteristic: Water depth: 1 (ft); Wave length: 5.2 (ft); Wave period: 1.1 (s); Wave height: 0.27 (ft) for the vertical force comparison and 0.2 (ft) for the horizontal force comparison. Plate geometry: Length (LP)= 2.5 ft (76.2 cm); Width=10 in (25.4 cm); Thickness(a)=1 7/8 in (4.7625 cm).

# References


Brater, E. F., McNown, J. S. & Stair, L. D. (1958), 'Wave forces on submerged structures', *Journal of the Hydraulics Division, ASCE* **84**, pp. 1–26.

Ertekin, R. C. (1984), Soliton Generation by Moving Disturbances in Shallow Water: Theory, Computation and Experiment, PhD thesis, Ph.D. Dissertation, university of California at Berkeley, May, v+352 pp. (TC 172.E74 1984).

Ertekin, R. C. & Becker, J. M. (1998), 'Nonlinear diffraction of waves by a submerged shelf in shallow water', *J. Offshore Mech. Arct. Eng.* **120**, pp. 212–220.

Ertekin, R. C., Webster, W. C. & Wehausen, J. V. (1986), 'Waves caused by a moving disturbance in a shallow channel of finite width', *Journal of Fluid Mechanics* **169**, pp. 275–292.

Green, A. E. & Naghdi, P. M. (1976), Directed fluid sheets, *in* 'Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences', Vol. 347, pp. 447–473.

Kaplan, P. (1992), Wave impact forces on offshore structures: Re-examination and new interpretations, *in* '24th Annual Conference on Offshore Technology', Houston, Texas, pp. 79–86.

Kim, J. W., Bai, K. J., Ertekin, R. C. & Webster, W. C. (2001), 'A derivation of the Green-Naghdi equations for irrotational flows', *Journal of Engineering Mathematics* **40**, pp. 17–42.

Morison, J. R., O'Brien, M. P., Johnson, J. W. & Schaaf, S. A. (1950), 'The force exerted by surface piles', *Petroleum Transactions* **Vol. 189**, 149–154.