Wave run-up on a vertical plate in an infinite wave field

by

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1 Introduction

The present study intends to trace the differences in the wave run-up on a plate, which is induced due to the incoming monochromatic waves in confined and infinite wave fields. The motivation of the present is the effect of the walls that was investigated and discussed in the works due to the senior author in [1-2]. In these studies both the mathematical formulation and evidently the experimental campaigns concerned a vertical plate experiencing the action of incoming waves in a confined field the transverse boundaries of which were univocally defined by the width of the basin. The main goal was to approach a condition where a ship structure is subjected to the action of heavy beam seas. It is evident that in reality relevant phenomena occur in open seas, which in view of applied mathematics are "infinite" wave fields.

In order to approach the details of the concerned situation, the plate herein is treated as an elliptical cylinder, fixed at infinite depth. The cylinder virtually becomes a rigid plate by letting its elliptic eccentricity approaching unity ($b/a \rightarrow 0$, where b and a are the semi-minor and semi-major axes). The method presented in the following accounts for the tertiary wave interactions introduced in [1] making the assumption that the wave field which impacts the plate is composed by the interaction of the reflected and the incoming waves.

2 An elliptical cylinder approximating a plate: the diffraction problem

The plate is formulated as an elliptical cylinder fixed at infinite water depth. The semi-minor axis is assumed to approximate zero which results in $u_0=\mathrm{atanh}(b/a)\rightarrow 0$ where u_0 denotes the elliptical boundary of the plate. The incident wave in elliptic coordinates, expressed in terms of infinite water depth, is written as [3]

$$\varphi_I = -2i\frac{g}{\omega}A_I e^{kz}\left\{\sum_{m=0}^{\infty} i^m \mathrm{Mc}_m^{(1)}(u;q)\mathrm{ce}_m(v;q)\mathrm{ce}_m(\alpha;q) + \sum_{m=1}^{\infty} i^m \mathrm{Ms}_m^{(1)}(u;q)\mathrm{se}_m(v;q)\mathrm{se}_m(\alpha;q)\right\} \tag{1}$$

where ω is the incident wave frequency, g is the gravitational acceleration, A_I is the wave amplitude, k is the wavelength, u and v are the elliptic coordinates, α is the angle of incidence, q is the Mathieu parameter $q=(ka\epsilon/2)^2$, ϵ is the elliptic eccentricity, ce_m and se_m are the even and odd periodic Mathieu functions and $\mathrm{Mc}_m^{(1)}$, $\mathrm{Ms}_m^{(1)}$ are the even and odd modified Mathieu functions of the first kind. A Cartesian coordinate system with axes x and y coinciding with the semi-major and semi-minor axes respectively requires that the angle of heading should be 90^o.

The diffraction potential which satisfy the bottom and free surface boundary conditions and the radiation condition at infinity is [3]

$$\varphi_D = -i\frac{g}{\omega}A_I e^{kz}\left\{\sum_{m=0}^{\infty} i^m B_m \mathrm{Mc}_m^{(3)}(u;q)\mathrm{ce}_m(v;q) + \sum_{m=1}^{\infty} i^m C_m \mathrm{Ms}_m^{(3)}(u;q)\mathrm{se}_m(v;q)\right\} \tag{2}$$

where $\mathrm{Mc}_m^{(3)}$ and $\mathrm{Ms}_m^{(3)}$ are the even and odd modified Mathieu functions of the third kind. In Eq. (2) B_m and C_m are unknown expansion coefficients which are obtained by applying the zero velocity condition on the elliptical boundary of the "plate". To this end the employment of the orthogonality relations of periodic Mathieu functions are required. Eventually the wave elevation in the incoming wave region is determined by $\eta = i\omega/g(\varphi_I + \varphi_D)$.

3 The parabolic equation

In Molin et al. [1] a parabolic equation is proposed that describes the transformation of the initially regular wave system under its tertiary interaction with the reflected wave field from the plate. The reflected wave system is locally idealized as a plane wave of amplitude a_R and direction β. A finite width b is given to the half-tank to solve for the space evolution of the complex amplitude of the incoming wave, starting from some distance ahead of the plate. The incoming wave amplitude is put under the form $A(x,y)=A_I(1+a(x,y))$ with $a(x,y)=\sum_{m=0}^{\infty} a_m(x)\cos(\lambda_m x)$, $\lambda_m=m\pi/b$ and the a_m coefficients satisfy the coupled evolution equations

$$\begin{aligned}&\frac{\mathrm{d}a_m(x)}{\mathrm{d}x}+\frac{i\lambda_m^2}{2k}a_m(x)-\frac{2ik\varepsilon^2}{(1+\delta_{0m})b}\sum_{n=0}^{\infty}\left\{\int_0^b\left[a_R^2(x,y)f(\beta)+1-\|1+a\|^2\right]\cos\lambda_m y\cos\lambda_n y\,\mathrm{d}y\right\}a_n(x)\\&=\frac{2ik\varepsilon^2}{(1+\delta_{0m})b}\int_0^b\left[a_R^2(x,y)f(\beta)+1-\|1+a\|^2\right]\cos\lambda_m y\,\mathrm{d}y\end{aligned}\tag{3}$$

Here $f(\beta)$ is a function of the angle of propagation of the reflected wave and has been given by Longuet-Higgins and Phillips [4].

In the works of Molin et al. [1-2] the reflected wave field is obtained via eigenfunction expansions based on the same width b of the domain. In the model developed here the reflected wave field corresponds to an infinite ocean and therefore is not seeing any wall!

For the purposes of the present contribution, the system (3) was solved by the Runge-Kutta method. In Eq. (3) δ denotes Kroneker's delta, ε is the wave steepness and the plane function a_R denotes the amplitude of the reflected wave, i.e. the equivalent plane wave, at any point (x,y) of the wave field.

Having calculated $A(x,y)$, a_R is obtained by employing the zero velocity condition on the plate and determining the unknown expansion coefficients B_m and C_m. These allow the derivation of the wave elevation due to the diffraction component in the entire wave field and accordingly its magnitude a_R. The wave elevation is given by

$$\begin{aligned}-\left(\frac{\pi}{2}\right)\frac{\eta_R}{A_I}&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} i^m\frac{\mathrm{Mc}_m^{(1)\prime}(u_0;q)}{\mathrm{Mc}_n^{(3)\prime}(u_0;q)}\mathrm{Mc}_n^{(3)}(u;q)\mathrm{ce}_n(v;q)\mathrm{ce}_m(\alpha;q)\int_0^{2\pi}A(u_0,v)\mathrm{ce}_m(v;q)\mathrm{ce}_n(v;q)\mathrm{d}v\\&+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} i^m\frac{\mathrm{Ms}_m^{(1)\prime}(u_0;q)}{\mathrm{Mc}_n^{(3)\prime}(u_0;q)}\mathrm{Mc}_n^{(3)}(u;q)\mathrm{ce}_n(v;q)\mathrm{se}_m(\alpha;q)\int_0^{2\pi}A(u_0,v)\mathrm{se}_m(v;q)\mathrm{ce}_n(v;q)\mathrm{d}v\\&+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} i^m\frac{\mathrm{Mc}_m^{(1)\prime}(u_0;q)}{\mathrm{Ms}_n^{(3)\prime}(u_0;q)}\mathrm{Ms}_n^{(3)}(u;q)\mathrm{se}_n(v;q)\mathrm{ce}_m(\alpha;q)\int_0^{2\pi}A(u_0,v)\mathrm{ce}_m(v;q)\mathrm{se}_n(v;q)\mathrm{d}v\\&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} i^m\frac{\mathrm{Ms}_m^{(1)\prime}(u_0;q)}{\mathrm{Ms}_n^{(3)\prime}(u_0;q)}\mathrm{Ms}_n^{(3)}(u;q)\mathrm{se}_n(v;q)\mathrm{se}_m(\alpha;q)\int_0^{2\pi}A(u_0,v)\mathrm{se}_m(v;q)\mathrm{se}_n(v;q)\mathrm{d}v\end{aligned}\tag{4}$$

The primes in Eq. (4) denote differentiation with respect to the argument. Also $A(u_0, v)$ is the incoming wave amplitude on the plate which for applying Eq. (4) is expressed in elliptic coordinates and it is taken at the elliptical boundary u_0 of the plate. Originally, the wave elevation is converted to elliptic coordinates as required by the integrands of Eq. (4). Next, the entire equivalent plane wave from Eq. (4) is converted to the (x,y) plane. The above procedure of successively updating the incoming and reflected wave systems is repeated as many times as are required to achieve convergence.

4 Numerical results

The numerical results presented in the following concern a 10m long plate installed in an infinite wave field and subjected to monochromatic incident waves of period 1.01s and wave steepness H/λ=2%, where λ is the wavelength and H is the wave height. The plate is formulated as an elliptical cylinder with an elliptic eccentricity

ϵ=0.9999 which results in u_0=0.0001. All calculations were performed using M=20 modes in the extended sums of Eqs. (1)-(3). The infinite wave field was extended transversely to -30m $\le y \le$30m and longitudinally to -$L \le x \le L$ with L being equal to 25.5m, 50m and 100m. Apparently, for the solution of the parabolic system the wave field on the quarter -$L \le x \le$0 and 0 $\le y \le$30m was employed.

Molin's et al. [1-2] method was applied iteratively at several steps without being known by default how many of them will be required eventually. Succinctly, each step starts with the solution of the parabolic system to calculate the wave elevation on the quarter of the field being employed, giving as an input the equivalent plane wave which is described by only the diffraction component. Next, the derived wave run-up on the plate is converted from Cartesian to elliptic coordinates in order to allow the use of Eq. (3) to determine the new equivalent plane wave in the entire (infinite) wave field. Originally, the equivalent plane wave is expressed in elliptic coordinates, as implied by Eq. (3), and accordingly it is converted to Cartesian. For the present model, satisfactory convergence was achieved after 9 iterations for L=25.5m and 50m and 20 iterations for the extended wave field of L=100m.

The basic findings of the present study are discussed with the aid of the following Figs. 1-3. Fig. 1 depicts the incoming wave amplitude ahead of the plate. As can be easily seen, the maximum elevation occurs exactly at the middle of the plate. The same figure shows also the results due to Molin et al. [2] who considered a confined wave field to simulate numerically their experimental set up. It is immediately apparent that the comparisons are very favorable. It is believed that the observed differences are due to the basic difference of the two configurations, namely here the plate is considered in an infinite wave field whereas in the works of Molin et al. [1-2] the field was specifically confined transversely. This remark is supported also by the wavy disturbances observed in [1] on the equivalent plane wave after plate's edge which in all probability are induced due to wave reflection by the walls of the numerical basin.

The total wave run-up is shown in Fig. 2 for several starting points of the incoming waves. Fig. 2 demonstrates an extreme amplification that reaches approximately four times the amplitude of the incident wave for the longer distance of L=100m. It is noted that the depicted results are those of the final iteration when the calculations have been converged. Indeed there are differences between the sequential iterations which imply that several recurrences of the interaction phenomenon are required to achieve a steady state condition. Also the results due to Molin et al. [2] are depicted and again the comparison is very favorable. As a conclusion, the wave train in the incoming wave field is provided and the associated results are shown in 3D in Fig. 3. The wave field is composed by the incoming and the reflected waves.

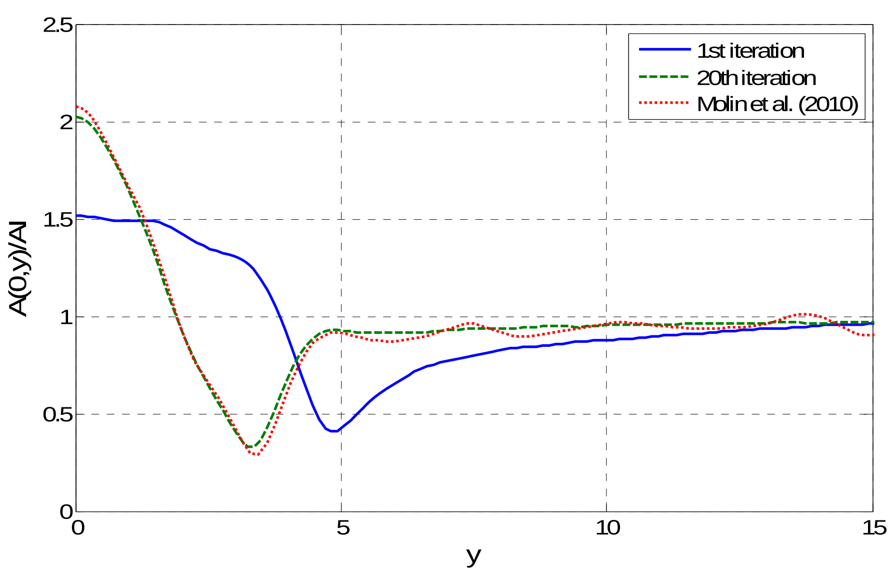

Fig.1. Incoming wave amplitude on the half plate at the end of iterations; waves start from x=-100m. The half plate extends from y=0 to y=5m.

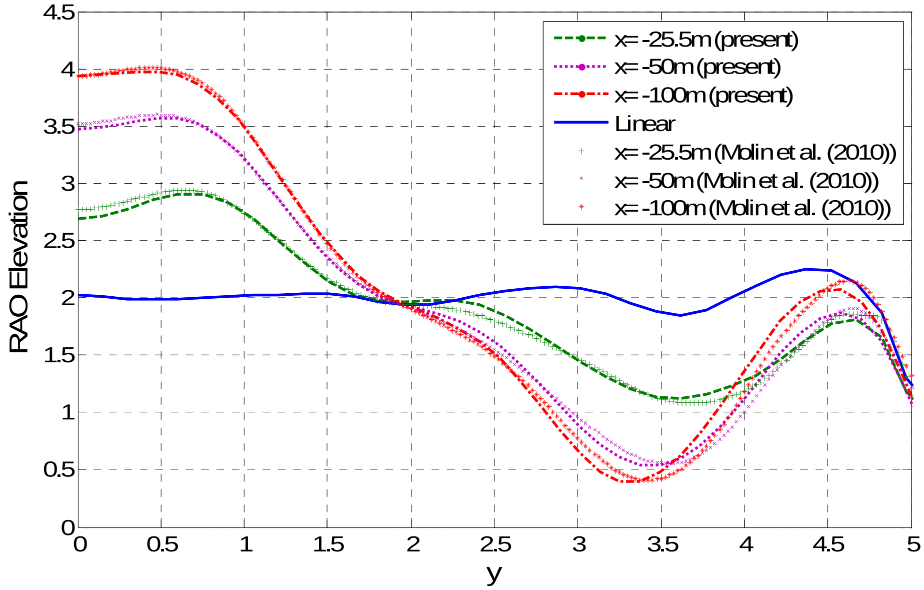

Fig. 2. Wave elevation on the plate for various wave starting points

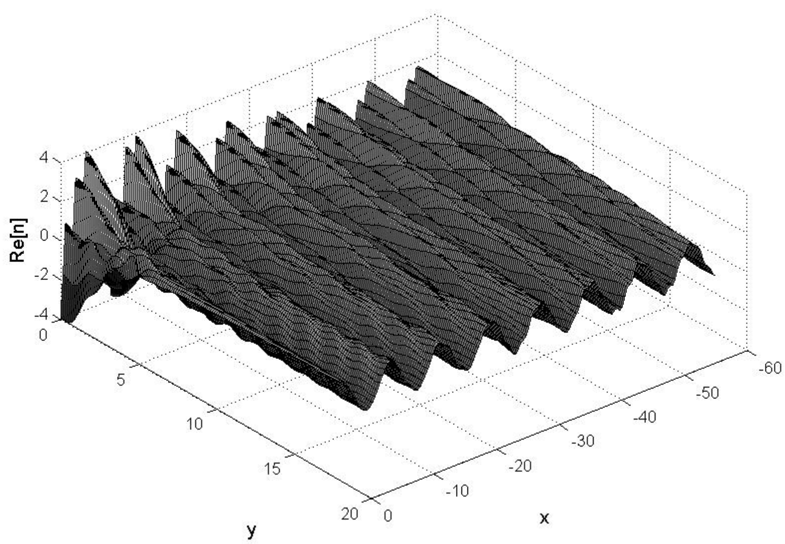

Fig. 3. Wave train at the end of iterations; waves start from *x*=-100m

5 References

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